VTC Training

Economic Engineering

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OUTLINE

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- Part 1: Cost Concepts and Design Economics
- Part 2: Cost Concepts and Design Economics
- Part 3: Time Value of Money

VTC Training

Part 1: Cost Concepts and Design Economics

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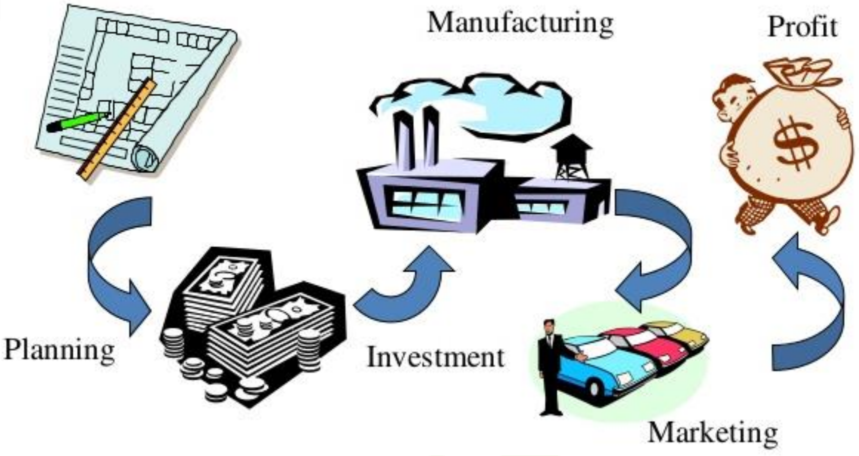
1.1 INTRODUCTION

- The objective of this training is to present the concepts and principles of engineering economy (EE).
- **EE involves the systematic evaluation of the merits of proposed solutions to engineering problems.**
- Engineers use knowledge to find new ways for doing things economically
- Engineering economy involves both economic and technical decisions
- An engineer who is unprepared to excel at engineering economy is not properly equipped for the job.



1.2 PRINCIPLES OF ENGINEERING ECONOMY

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1. Develop the alternatives

Carefully define the problem! Then the choice (decision) is among alternatives. The alternatives need to be identified and then defined for subsequent analysis.

2. Focus on the differences

Only the differences in expected future outcomes among the alternatives are relevant to their comparison and should be considered in the decision

3. Use a consistent viewpoint

The prospective (مستقبلي) outcomes of the alternatives, economic and other, should be consistently developed form a defined viewpoint (perspective).

4. Use a common unit of measure

Using a common unit of measurement to list as many of the prospective outcomes as possible will simplify the analysis of the alternatives.

1.2 PRINCIPLES OF ENGINEERING ECONOMY

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5. Consider all relative criteria

Selection of a preferred alternative requires the use of a criterion/criteria. The decision process should consider both the outcomes listed in the monetary unit and those expressed in some other unit of measurement in a descriptive manner.

6. Make risk and uncertainty explicit

Risk and uncertainty are inherent in estimating the future outcomes of the alternatives and should be recognized in their analysis and comparison.

7. Revisit your decisions

Improved decision making results from an adaptive process; to the extent practicable, the initial projected outcomes of the selected alternative should be subsequently compared with actual results achieved.

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Steps of EE Analysis Procedure	Activities of Engineering Design Process
1. Problem recognition, definition, and evaluation	1. Problem/need definition
2. Development of the feasible alternatives	2. Problem/need formulation and evaluation.
3. Development of the outcomes and cash flows for each alternative	3. Synthesis of possible solutions (alternatives)
4. Selection of a criterion/criteria	4. Analysis, optimization, and evaluation.
5. Analysis and comparison of the alternatives	5. Specification of preferred alternative.
6. Selection of the preferred alternative	6. Communication
7. Performance monitoring and post-evaluation of results	

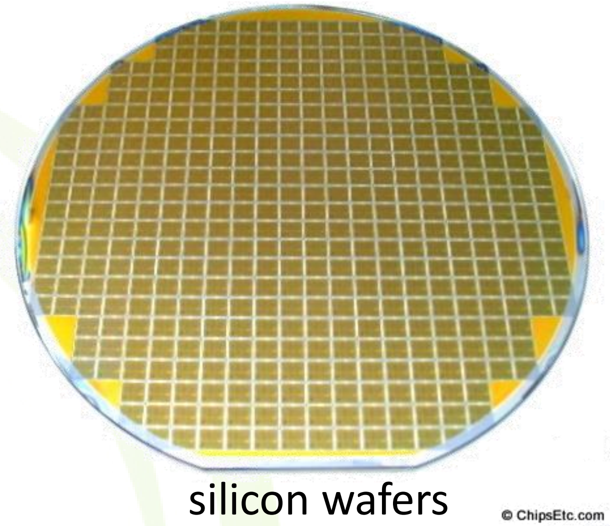
1.3 ENGINEERING ECONOMY & THE DESIGN PROCESS

Ex 1-1: Defining the problem and developing alternatives

A solar cell manufacturer buys silicon wafers (رقائق السليكون) and converts them into solar cells, to be used to generate power in solar panels. Cell manufacturers faced a sharp decline in the price of their product in 2011-2012. Facing declining profitability, the manufacturer considers two solutions: introducing measures to reduce wastage in production, and salvaging cells that are damaged in production in order to sell them to toy manufacturers.

a) Define the company’s problem. Next, reformulate the problem in a creative way

b) Evaluate the proposed solutions and discuss how they can address your reformulated problem. (don’t concern yourself with feasibility as this point)



silicon wafers

solar cells

Ex 1-1: Defining the problem and developing alternatives

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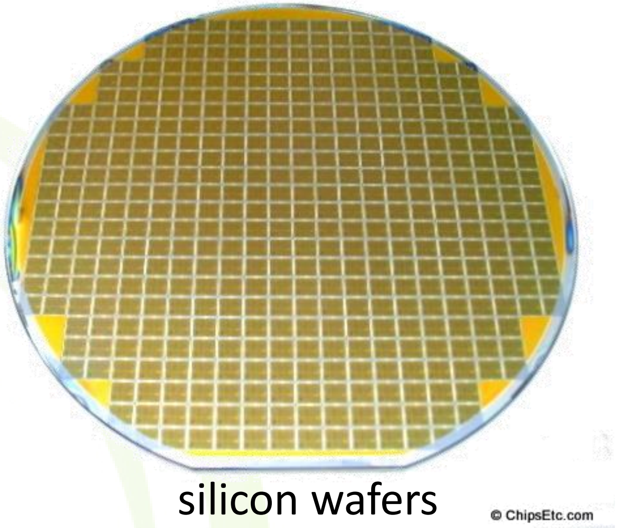
a) Define the company's problem. Next, reformulate the problem in a creative way

Problem: decreased revenues leading to a decline in profitability.

Several reformulations can be posed:

1. Find methods to reduce costs below the new revenues
2. Find avenues to increase revenues to cover existing costs
3. Evaluate whether the business continues to be viable, or whether it would be wiser to exist in the industry altogether.



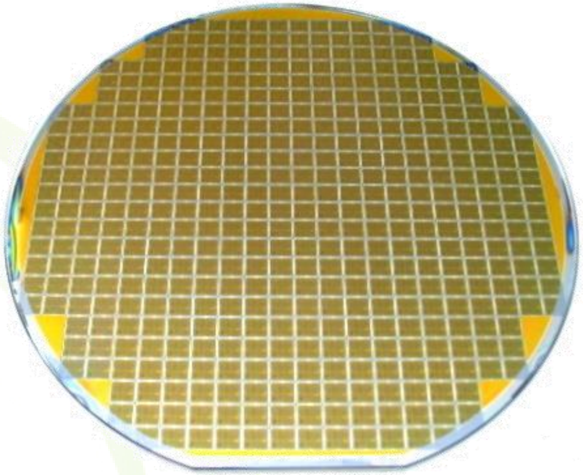
silicon wafers

solar cells

Ex 1-1: Defining the problem and developing alternatives

b) Evaluate the proposed solutions and discuss how they can address your reformulated problem. (don't concern yourself with feasibility as this point)

1. first formulation – reducing costs while maintaining the new level of reduced revenues.
   - The firm could increase production floor efficiencies by reducing waste and reusing raw materials as much as possible.
2. Second solution – increasing revenues by introducing a new production in low-end solar products that are sold to toy manufacturers and made from waste materials.
3. third solution - could be renegotiating contracts with suppliers of the silicon wafers.



silicon wafers

solar cells

Developing investment alternatives: ideas generation

Classical brainstorming

- Rules:
  - Criticism is ruled out
  - Freewheeling (لا قيود) is welcomed
  - Quantity is wanted
  - Combination and improvement are sought
- Procedure:
  - Preparation: participants, problem statement
  - Brainstorming
  - Evaluation

Nominal group technique (NGT)

- Advantages:
  - Reduce the dominance of one/more participants
  - Reduce suppression of conflicting ideas
- Format
  - Individual silent generation of ideas
  - Individual round-robin feedback and recording of ideas
  - Group clarification of each idea
  - Individual voting and ranking to prioritize ideas
  - Discuss of group consensus results

Ex 1-2: Application of the EE analysis procedure

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Your friend is considering investing in a wo-year MBA program. Tuition costs will be $60,000 for two years while living expenses will be $25,000 per year. She has $10,000 in savings, which she can spend on her education, and will need to borrow the rest from her bank. Her annual loan repayment will be $10,500. she currently works as an analyst and makes $60,000 a year; after she gets her degree she hopes to work as a manager for $150,000 a year.

Refer to the seven-step procedure in table 1-1 (left side) to answer these questions:

a) How should your friend formulate her problem?

b) What are her all projected costs ?

c) Suggest alternatives to your friend to reduce the uncertainty associated with finding a high-income job to pay off her loan

d) Select a criterion for discriminating among alternatives, and use it to advise your friend on which course of action to pursue

e) Attempt to analyze and compare the alternatives in view of at least one criterion in addition to cost

f) What should your friend do based on the information you and she have generated?

Steps of EE Analysis Procedure
1. Problem recognition, definition, and evaluation
2. Development of the feasible alternatives
3. Development of the outcomes and cash flows for each alternative
4. Selection of a criterion/criteria
5. Analysis and comparison of the alternatives
6. Selection of the preferred alternative
7. Performance monitoring and post-evaluation of results

1.3 ENGINEERING ECONOMY & THE DESIGN PROCESS

Ex 1-2: Application of the EE analysis procedure

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a) How should your friend formulate her problem?

- Decision: whether or not to invest in an MBA program.
- Pros: a higher income stream for the rest of her working life
- Cons: cannot be certain of getting her dream job.
- Problem: the risk she takes on a large loan of $100,000 to finance the MBA, but cannot find a job afterwards that allows her to pay it off.

Steps of EE Analysis Procedure

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b) What are her all projected costs ?

Cost	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Total
Tuition	-$30,000	-$30,000									-$60,000
Living	-$25,000	-$25,000									-$50,000
Net loan payment	-$500	-$500	-$500	-$500	-$500	-$500	-$500	-$500	-$500	-$500	-$5,000
Loss earnings	-$60,000	-$60,000									-$120,000
Total	-$115,500	-$115,500	-$500	-$500	-$500	-$500	-$500	-$500	-$500	-$500	-$235,000

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c) Suggest alternatives to your friend to reduce the uncertainty associated with finding a high-income job to pay off her loan

#	Option	Pros	Cons
1	Lower cost degree, part time MBA	• Work during study • Reduce direct/indirect costs • Reduce debt amount	Lower expected income after the degree
2	Postpone degree to save money for it	• Reduce loan amount • Reduce interest cost	Continue to earn at a lower level till she completes MBA
3	Not to do MBA	• No cost	No income imporvement

1.3 ENGINEERING ECONOMY & THE DESIGN PROCESS

Ex 1-2: Application of the EE analysis procedure

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d) Select a criterion for discriminating among alternatives, and use it to advise your friend on which course of action to pursue.

Criterion: minimize risk

Option: second or third as they involve no risk

Steps of EE Analysis Procedure

1. Problem recognition, definition, and evaluation
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e) Attempt to analyze and compare the alternatives in view of at least one criterion in addition to cost

Credit worthiness (الجدارة الائتمانية): a valuation performed by lenders that determines the possibility a borrower may default on his debt obligations.

Taking on any kind of debt without a guaranteed job a the end of MBA would be ruled out.

Hence option 3 may be her only option.

Steps of EE Analysis Procedure

1. Problem recognition, definition, and evaluation
2. Development of the feasible alternatives
3. Development of the outcomes and cash flows for each alternative
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1.3 ENGINEERING ECONOMY & THE DESIGN PROCESS

Ex 1-2: Application of the EE analysis procedure

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f) What should your friend do based on the information you and she have generated?

To reduce uncertainty regarding employment, she should:

- Gather information on the state of the job market in both the geographical area and the sector
- Find out which MBA programs are particularly strong at driving recruitment
- Try to negotiate a job with a desired employer before starting the MBA program

Steps of EE Analysis Procedure

1. Problem recognition, definition, and evaluation
2. Development of the feasible alternatives
3. Development of the outcomes and cash flows for each alternative
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Ex 1-3: Get rid of the old clunker?

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Linda and Jerry are faced with a car replacement opportunity where an interest rate can be ignored. Jerry's old clunker that averages 10 mpg of gasoline can be traded in toward a vehicle that gets 15 mpg. Or, Linda's 25 mpg car can be traded in toward a new hybrid vehicle that averages 50 mpg.

If they drive both cars 12,000 miles per year and their goal is to minimize annual gas consumption, which car should be replaced? They can only afford to upgrade on car at this time.

Jerry's old clunker
10 mpg = 4.25 km/l

Linda's car
25 mpg = 10.6 km/l

	Jerry	Linda
annual distance (mile/y)	12,000	12,000
average gas consumption (mpg)	10	25
car -gas consumption (gallon/y)	1,200	480
traded vehicle gas consumption (mpg)	15	50
total car -gas consumption (gallon/y)	800	240
saved gas consumption (gallon/y)	400	240

1.4 SPREADSHEETS IN EE ANALYSIS

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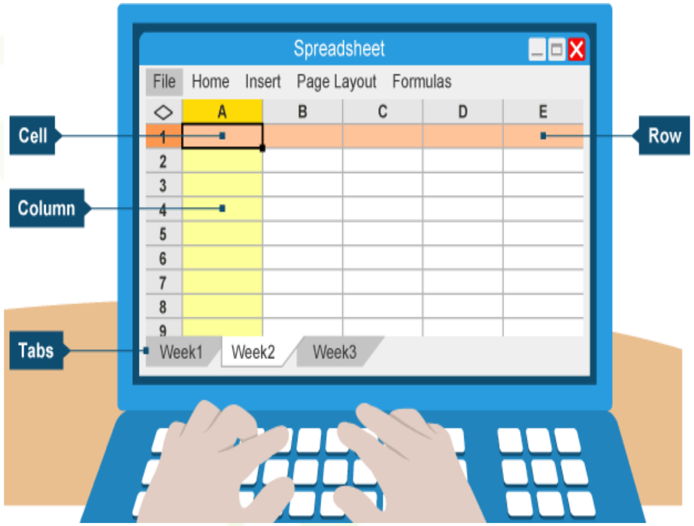
- Spreadsheets are a useful tool for solving EE problems.
- Reasons:
  1. Consist of calculations expressed as formulas rely on a few functional relationships
  2. Problem parameters are subject to change
  3. Results and calculations must be documented
  4. Graphical output is often required.



END OF PART 1

"Engineering is the conscious application of science to the problems of economic production."

- H. P. Gillette (1910)

VTC Training

Part 2: Cost Concepts and Design Economics

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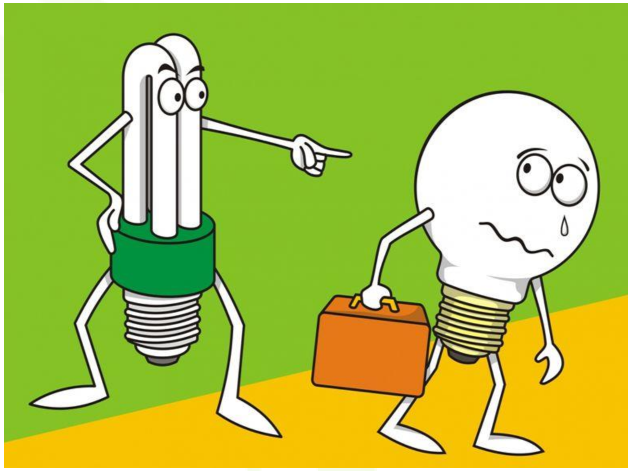
1.1 INTRODUCTION

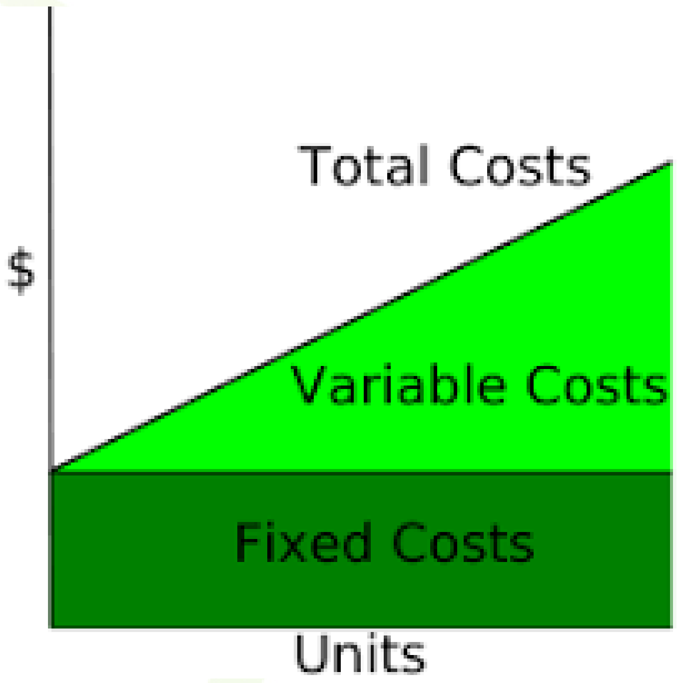
- The objective is to analyze short-term alternatives when the time value of money is not a factor.
- This is accomplished with three types of problems:
    1. Economic breakeven analysis
    2. Cost-driven design optimization
    3. Present economy studies



2.1 COST TERMINOLOGY

2.1.1 Fixed, Variable, and Incremental Costs

- **Fixed cost**: unaffected by changes in activity level:
  - Insurance and taxes on facilities
  - General management and administrative salaries
  - License fees
  - Interest costs on borrowed capital
- **Variable cost**: vary in total with the quantity of output (or similar measure of activity)
  - Cost of raw materials
  - Cost of labor used in a product service
- **Incremental cost**: additional cost resulting from increasing output of a system by one (or more) units
  - Associated with "go-no go" decision involved a limited change in output or activity level
  - Difficult to determine in practice



Example 2-1 Fixed and variable costs

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In connection with surfacing a new highway, a contractor has a choice of two sites on which to set up the asphalt-mixing plant equipment. The contractor estimates that it will cost $2.75 per cubic yard ($yd^3$-mile) to haul (drag) the asphalt-paving material from the mixing plant to the job location. Factors relating to the two mixing sites are as follows (production costs at each site are the same):

Cost factor	Site A	Site B
Average hauling distance	4 miles	3 miles
Monthly rental of site	$2,000	$7,000
Cost to setup and remove equipment	$15,000	$50,000
Hauling expense	$2.75/$yd^3$-mile	$2.75/$yd^3$-mile
Flag-person	Not required	$150/day

The job required 50,000 cubic yards of mixed-asphalt-paving material. It is estimated that four months (17 weeks of five working days per week) will be required for the job. Compare the two sites in terms of their fixed, variable, and total costs.

a) Which is the better site?

b) For the selected site, how many cubic yards of paving material does the contractor have to deliver before starting to make a profit if paid $12 per cubic yard delivered to the job?

Example 2-1 Fixed and variable costs Solution

The fixed and variable costs for this job are indicated in the table shown next

Fixed costs: site rental, setup, and removal costs (and flag-person cost at site B)

Variable costs: hauling cost

Cost	Fixed	Variable	Site A	Site B
Rental	X		=4months X \$2,00/month **=\$8,000**	=4months X \$7,00/month **=\$28,000**
Setup / removal	X		**=\$15,000**	**=\$50,000**
flagperson	X		=0	=5d/w X 17 W/period X\$150/d =\$12,750
Hauling		X	=4 mile X 50,000 yd^3 X \$2.75/yd^3-mile **=\$550,000**	=3 mile X 50,000 yd^3 X \$2.75/yd^3-mile **=\$412,000**
Total Cost			**=\$573,000**	**=\$503,250**

a) Site B has the smaller total cost!

Example 2-1 Fixed and variable costs Solution

Cost	Fixed	Variable	Site A	Site B
Rental	X		=4months X \$2,00/month =**\$8,000**	=4months X \$7,00/month **=\$28,000**
Setup / removal	X		**=\$15,000**	**=\$50,000**
flagperson	X		=0	=5d/w X 17 W/period X\$150/d =\$12,750
Hauling		X	=4 mile X 50,000 yd^3 X \$2.75/yd^3-mile **=\$550,000**	=3 mile X 50,000 yd^3 X \$2.75/yd^3-mile **=\$412,000**
Total Cost			**=\$573,000**	**=\$503,250**

b) Profit begin at the point: **total revenue = total cost** (as function of cubic yards asphalt delivered)

Based on site B:

Variable cost: 3 miles X \$2.75/yd^3-mile = \$8.25/ yd^3 delivered

Fixed cost: \$28,000 + \$50,000 + \$12,750 = \$90,750

Total cost = Fixed + Variable = Revenue

\$90,750 + \$8.25 Y = \$12Y

Y=24,200 yd^3 delivered

2.1.2 Direct, Indirect, and Standard Costs (We need to use common cost terminology)

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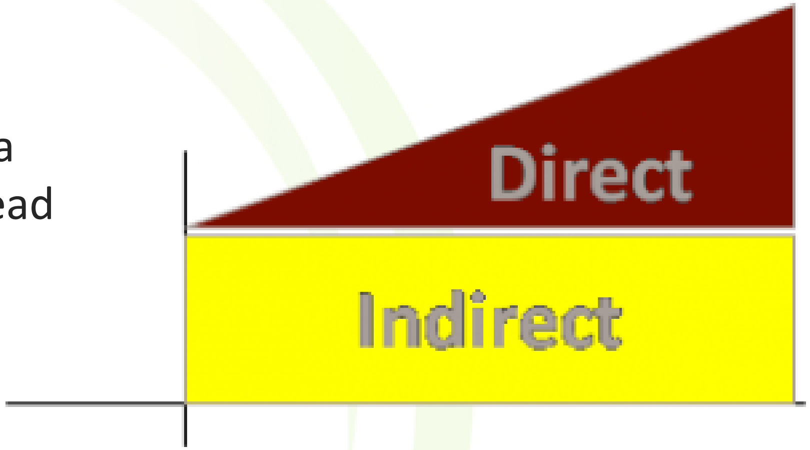
- **Direct**: can be measured and allocated to a specific work activity
  - Labor and material costs
  - Construction activity
- **Indirect**: difficult to attribute or allocate to a specific output or work activity (also overhead or burden)
  - Cost of common tools
  - General supplies
  - Equipment maintenance
  - Overhead: electricity, general repairs, property taxes, supervision
- **Standard** : cost per unit of output, established in advance of production or service delivery



2.1 COST TERMINOLOGY

(We need to use common cost terminology)

- **Cash cost**: a cost that involves a payment of cash.

- **Book cost**: a cost that does not involve a cash transaction but is reflected in the accounting system (noncash cost).
  - Depreciation

- **Sunk cost**: a cost that has occurred in the past and has no relevance to estimates of future costs and revenues related to an alternative course of action.

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2.1 COST TERMINOLOGY

(more common cost terminology)

- **Opportunity cost**: the potential benefit that is given up due to seeking an alternative course of action.
  - A student earns $25,000 per year
  - Choose to go to school for a year and spend $5,000
  - Opp. Cost = $25,000

2.1 COST TERMINOLOGY

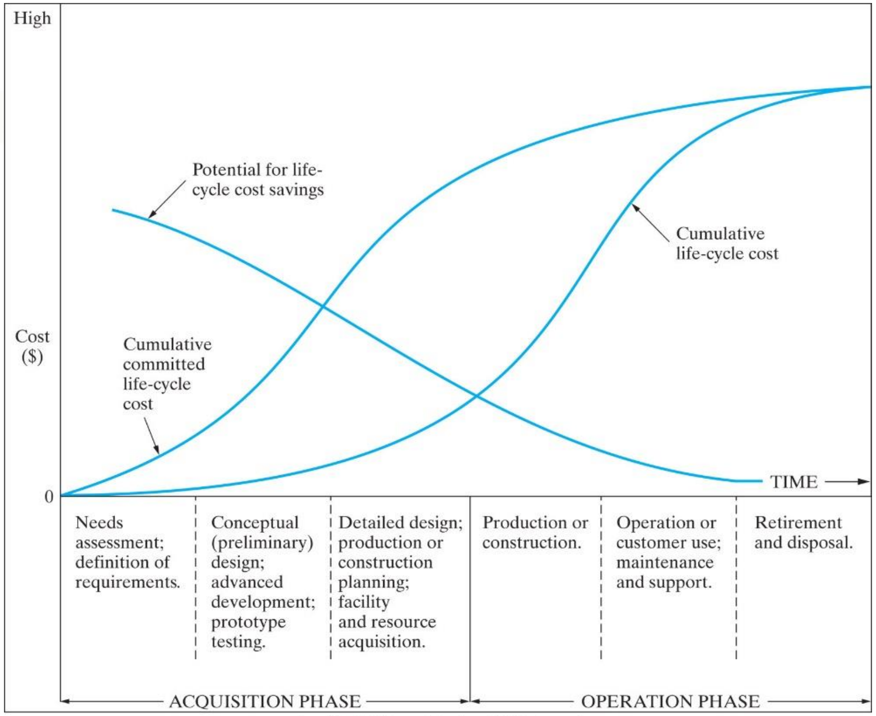
(more common cost terminology)

Life-cycle cost: the summation of all costs related to a product, structure, system, or service during its life span.

Investment cost: the capital required for most of the activities in the acquisition phase

Operation and maintenance cost (O&M): associated with operation phase

Disposal Costs: nonrecurring costs of shutdown, retirement and disposal of assets



2.2 THE GENERAL ECONOMIC ENVIRONMENT

Consumer & producer goods & services

- Economics deals with interactions between people and wealth
- Engineering is concerned with the cost-effective use of scientific knowledge to benefit humankind.
- Goods and services may be divided into two classes

1. **Consumer goods & services:**
   - those products or services that are directly used by people to satisfy their wants
   - Examples: food, clothing, homes, cars, TVs, medical services…

2. **Producer goods and services:**
   - Used to produce consumer goods and services or other producer goods
   - Examples: machine tools, factory buildings, buses, and farm machinery

- The amount of producer goods needed is determined indirectly by the amount of consumer goods or services that are demanded by people.

Measures of economic worth

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- Goods and services are produced and desired because they have **utility** – the power to satisfy human wants and needs
- Utility is most commonly measured in terms of **value**, expressed as the **price**
- Much of the business activity, including engineering, focuses on increasing the utility (value) of materials and products by changing their form or location
- Ex: iron ore, worth only a few dollars per ton, significantly increases in value by being processed.

Necessities, luxuries, and price demand

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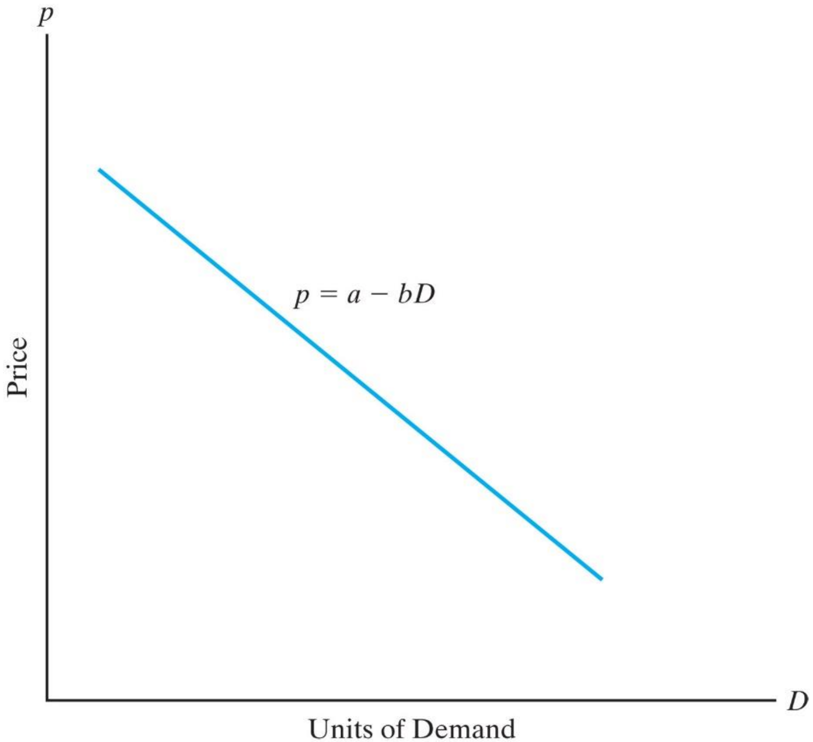
- Goods and services may be divided into two types: necessities and luxuries
- There is a relationship between the price and the quantity:
- $P = a - bD$

  $$for\ 0 \le D \le \frac{a}{b}, \qquad and\ a > 0, b > 0$$

- $D = \frac{a-p}{b} \quad (b \neq 0)$



2.2 THE GENERAL ECONOMIC ENVIRONMENT

Competition

- Most general economic principles are stated for situations in which perfect competition exists.
- **Perfect competition** occurs when:
  - a product is supplied by a large number of vendors
  - no restriction on additional suppliers entering the market.
- **Monopoly**: the opposite of perfect competition and exists when
  - a unique product or service is only available from a single supplier
  - vendor can prevent the entry of all others into the market

2.2 THE GENERAL ECONOMIC ENVIRONMENT

The total revenue function

- $TR = price\ x\ demand = p \cdot D$
- $TR = (a - bD)D = aD - bD^2$

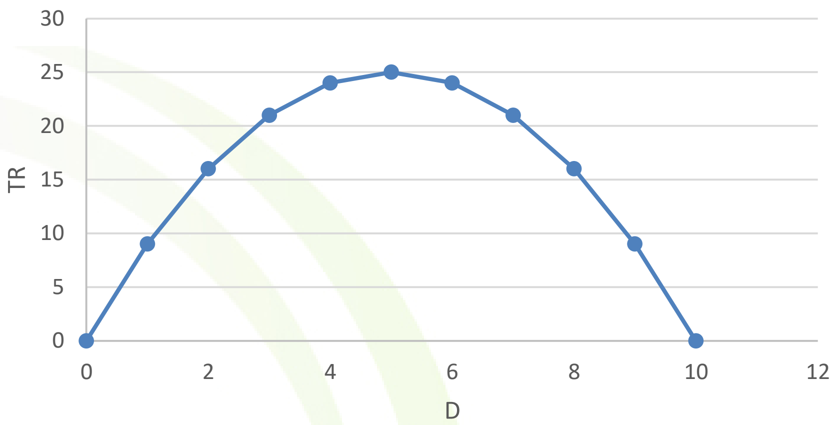
$$for\ 0 \leq D \leq \frac{a}{b}\ \ and\ a > 0, b > 0$$

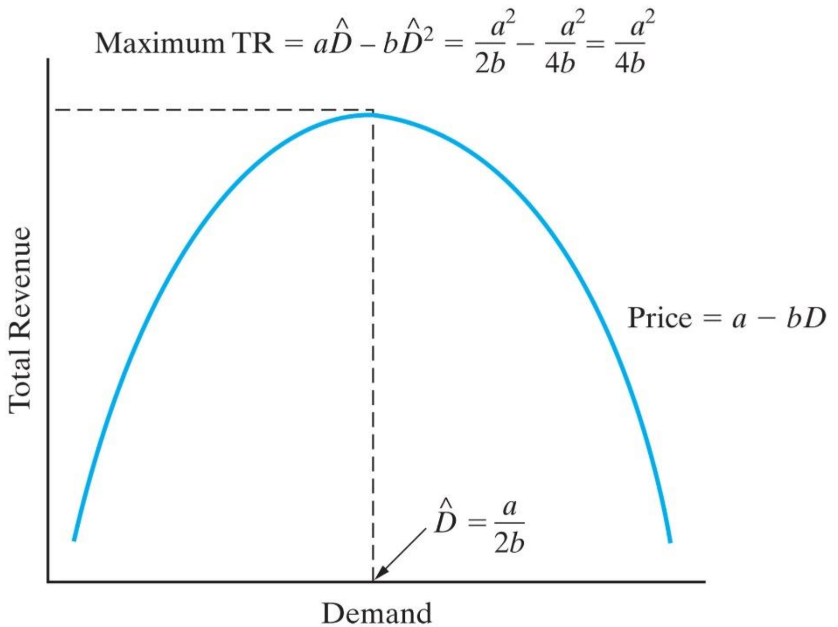
- The demand for max. revenue

$$\frac{dTR}{dD} = a - 2bD = 0$$

$$\hat{D} = \frac{a}{2b}$$

- Most businesses would not obtain max. profits by maximizing revenue.





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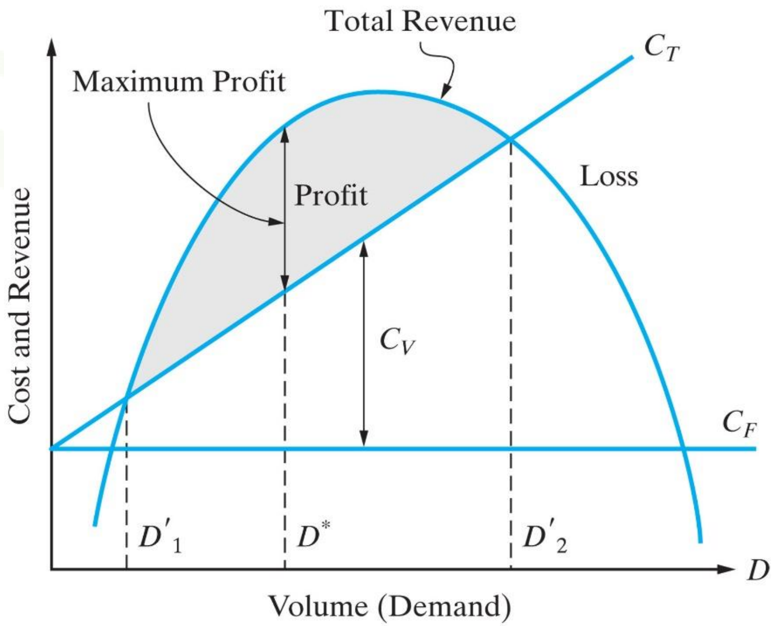
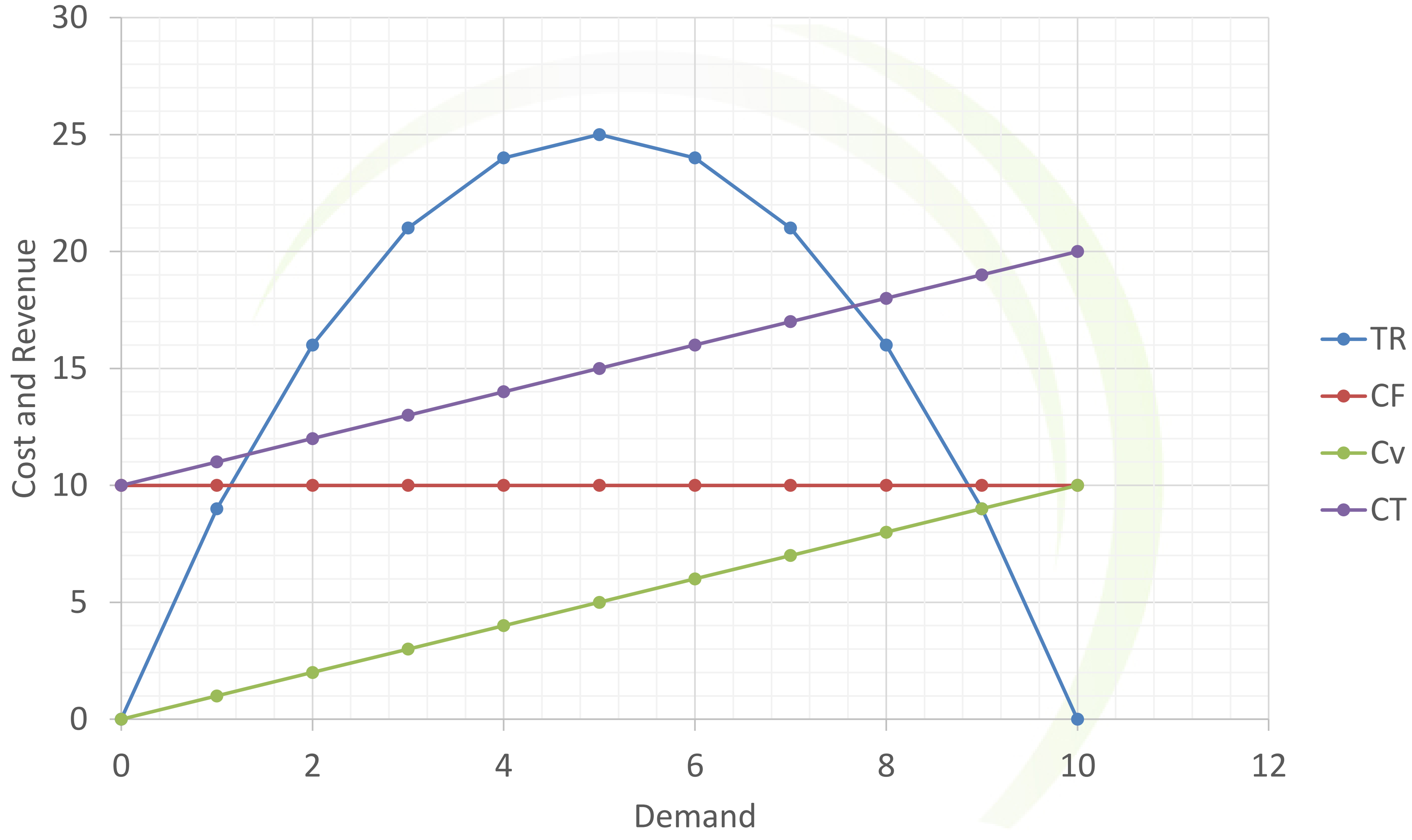
Cost, volume, and breakeven point relationships

- Fixed costs (C_F)remain constant over a wide range of activities
- Variable costs (C_V) vary with volume (demand)
  - c_v : variable cost per unit
- Total cost (C_T) is the sum of both
- $C_T = C_F + C_V$
- $C_V = c_v \cdot D$
- Consider two scenarios for finding breakdown analysis




Cost and Revenue
30
25
20
15
10
5
0
0
2
4
6
8
10
12
Demand
TR
CF
Cv
CT

Cost, volume, and breakeven point relationships

Scenario 1: price dependent on demand

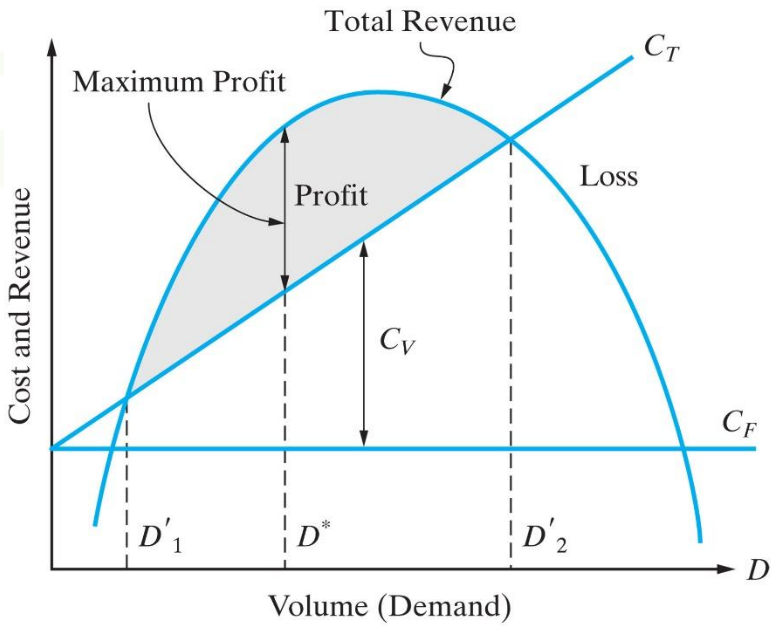
- $C_T = C_F + C_V$
- $C_V = c_v \cdot D$
- $C_T = C_F + c_v \cdot D$
- Breakeven points:
  - D'_1 : $total\ revenue = total\ cost$
  - D^* : $optimal\ D, at\ max. profit$
  - D'_2 : $total\ revenue = total\ cost$
- $Profit(loss) = total\ revenue - total\ cost$

$$= (aD - bD^2) - (C_F + c_v \cdot D)$$

$$= -bD^2 + (a - c_v)D - C_F$$

$$for\ 0 \leq D \leq \frac{a}{b}\ \ and\ a > 0, b > 0$$

- Profit occurs:
  - $a - c_v > 0$
  - $TR > C_T$



Cost, volume, and breakeven point relationships

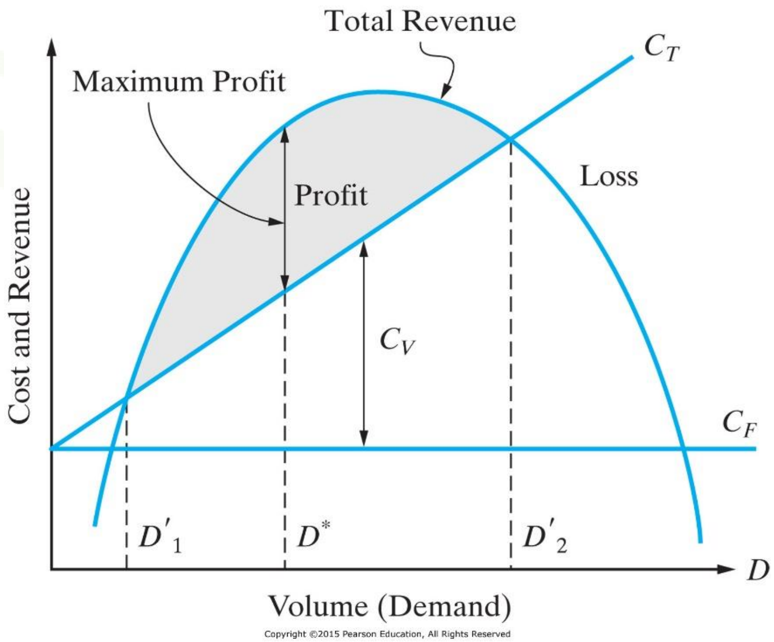
Scenario 1

- $Optimal\ demand$:

$$\frac{d(profit)}{dD} = a - c_v - 2bD = 0$$

$$D^* = \frac{a - c_v}{2b}$$

- For max profit(not min), the sign of the 2nd derivative must be negative
- $\frac{d^2(profit)}{dD^2} = -2b.$ Negative for b>0
- TR=TC. (breakeven point)
- $aD - bD^2 = C_F + c_v \cdot D$
- $= -bD^2 + (a - c_v)D - C_F = 0$
- Quadratic eq, solve for D at breakeven points D_1', D_2'
- $D' = \frac{-(a-c_v) \pm [(a-c_v)^2 - 4(-b)(-C_F)]^{1/2}}{2(-b)}$



Example 2-4 optimal demand when function of price

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A company produces as an electronic timing switch that is used in consumer and commercial products.

The fixed cost (C_F) is \$73,000 per month

The variable cost (c_v) is \$83 per unit

The selling price per unit is $p = \$180 - 0.02D$

a) Determine the optimal volume for this product and confirm that a profit occurs (instead of a loss) at this demand

b) Find the volumes at which breakeven occurs, that is, what is the range of profitable demand? Solve by hand and spreadsheet.

Example 2-4 optimal demand when function of price

A company produces as an electronic timing switch that is used in consumer and commercial products.

The fixed cost (C_F) is \$73,000 per month

The variable cost (c_v) is \$83 per unit

The selling price per unit is $p = \$180 - 0.02D$

a) Determine the optimal volume for this product and confirm that a profit occurs (instead of a loss) at this demand

$$D^* = \frac{a - c_v}{2b} = \frac{\$180 - \$83}{2(0.02)} = 2{,}425\ units\ per\ month$$

Is $a - c_v > 0$? $\$180 - \$83) = \$97 > 0$. yes

Is (total revenue-total cost)>0 for $D^* = 2{,}425\ units\ per\ month$?

$[\$180(2{,}425) - 0.02(2{,}425)^2] - [\$73{,}000 + \$83(2{,}425)] = \$44{,}612$

A demand of $D^* = 2{,}425$ units per month results in a max profit of \$44,612 per month. Notice the 2nd derivative is negative (-.04)

Example 2-4 optimal demand when function of price

The fixed cost (C_F) is \$73,000 per month
The variable cost (c_v) is \$83 per unit
The selling price per unit is $p = \$180 - 0.02D$

b) Find the volumes at which breakeven occurs, that is, what is the range of profitable demand? Solve by hand and spreadsheet.

- TR=TC. (breakeven point)
- $-bD^2 + (a - c_v)D - C_F = 0$
- $-0.02D^2 + (\$180 - \$83)D - \$73{,}000 = 0$
- $-0.02D^2 + 97D - \$73{,}000 = 0$
- $D' = \frac{-(a-c_v)\pm[(a-c_v)^2-4(-b)(-C_F)]^{1/2}}{2(-b)}$
- $D' = \frac{-(97)\pm[(97)^2-4(-0.02)(-73{,}000)]^{1/2}}{2(-0.02)}$
  - $D_1' = 932\ units\ per\ month$
  - $D_2' = 3{,}918\ units\ per\ month$
- The range of profitable demand is 932-3,918 units per month

Example 2-4 optimal demand when function of price

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= \$B\$3 – \$B\$4 * A7

= \$B\$1 + \$B\$2 * A7

= B7 * A7

= C7 – D7

= E1

= A7 + \$E\$2

	A	B	C	D	E
1	Fixed cost/ mo. =	\$ 73,000		Demand Start point (D) =	0
2	Variable cost/unit =	\$ 83		Demand Increment =	250
3	a =	\$ 180			
4	b =	\$ 0.02			
5					
6	**Monthly Demand**	**Price per Unit**	**Total Revenue**	**Total Expense**	**Profit**
7	0	\$ 180	\$ -	\$ 73,000	\$ (73,000)
8	250	\$ 175	\$ 43,750	\$ 93,750	\$ (50,000)
9	500	\$ 170	\$ 85,000	\$ 114,500	\$ (29,500)
10	750	\$ 165	\$ 123,750	\$ 135,250	\$ (11,500)
11	1000	\$ 160	\$ 160,000	\$ 156,000	\$ 4,000
12	1250	\$ 155	\$ 193,750	\$ 176,750	\$ 17,000
13	1500	\$ 150	\$ 225,000	\$ 197,500	\$ 27,500
14	1750	\$ 145	\$ 253,750	\$ 218,250	\$ 35,500
15	2000	\$ 140	\$ 280,000	\$ 239,000	\$ 41,000
16	2250	\$ 135	\$ 303,750	\$ 259,750	\$ 44,000
17	2500	\$ 130	\$ 325,000	\$ 280,500	\$ 44,500
18	2750	\$ 125	\$ 343,750	\$ 301,250	\$ 42,500
19	3000	\$ 120	\$ 360,000	\$ 322,000	\$ 38,000
20	3250	\$ 115	\$ 373,750	\$ 342,750	\$ 31,000
21	3500	\$ 110	\$ 385,000	\$ 363,500	\$ 21,500
22	3750	\$ 105	\$ 393,750	\$ 384,250	\$ 9,500
23	4000	\$ 100	\$ 400,000	\$ 405,000	\$ (5,000)
24	4250	\$ 95	\$ 403,750	\$ 425,750	\$ (22,000)
25	4500	\$ 90	\$ 405,000	\$ 446,500	\$ (41,500)
26	4750	\$ 85	\$ 403,750	\$ 467,250	\$ (63,500)
27	5000	\$ 80	\$ 400,000	\$ 488,000	\$ (88,000)
28	5250	\$ 75	\$ 393,750	\$ 508,750	\$ (115,000)
29	5500	\$ 70	\$ 385,000	\$ 529,500	\$ (144,500)

(a) Table of profit values for a range of demand values

2.2 THE GENERAL ECONOMIC ENVIRONMENT

Example 2-4 optimal demand when function of price

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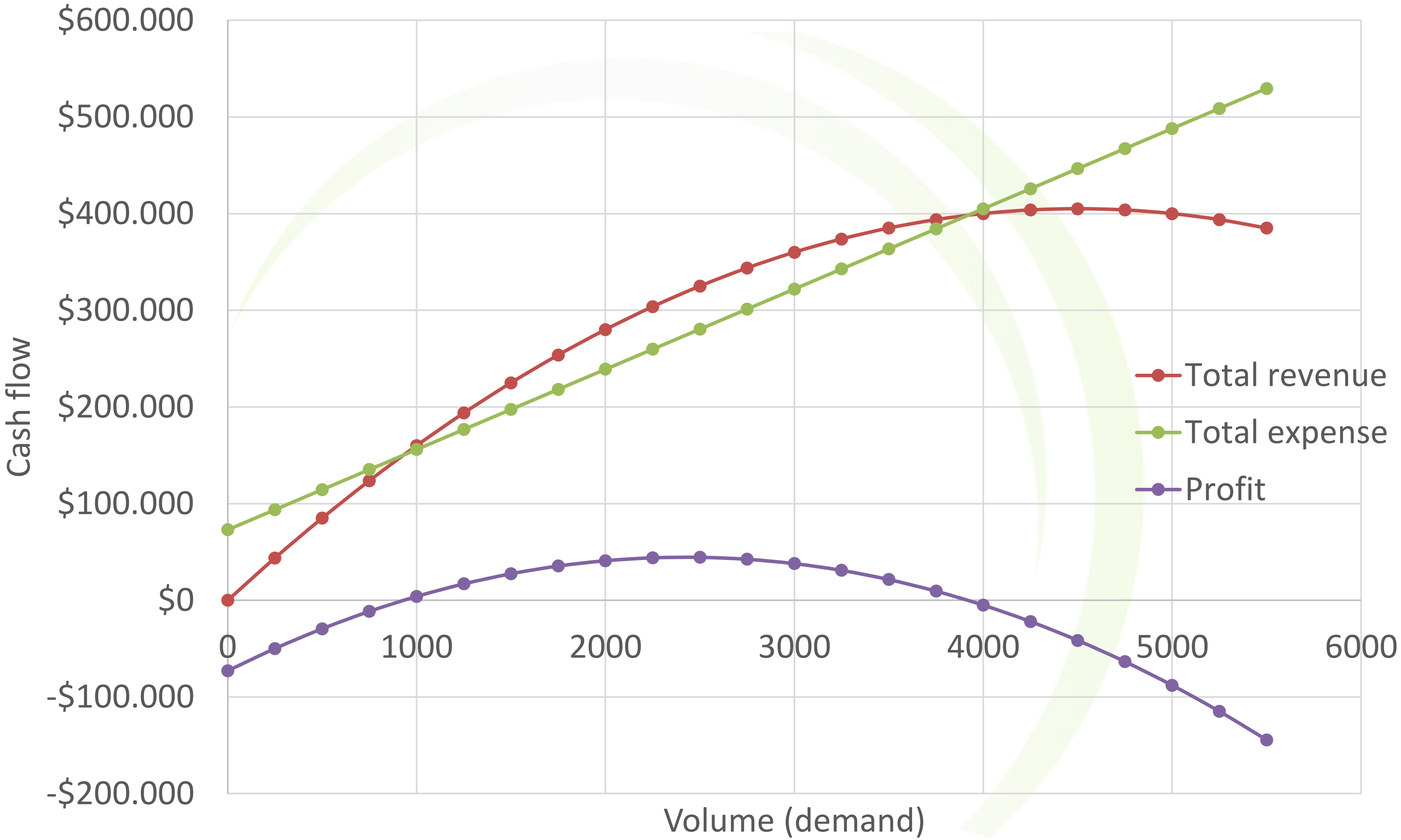

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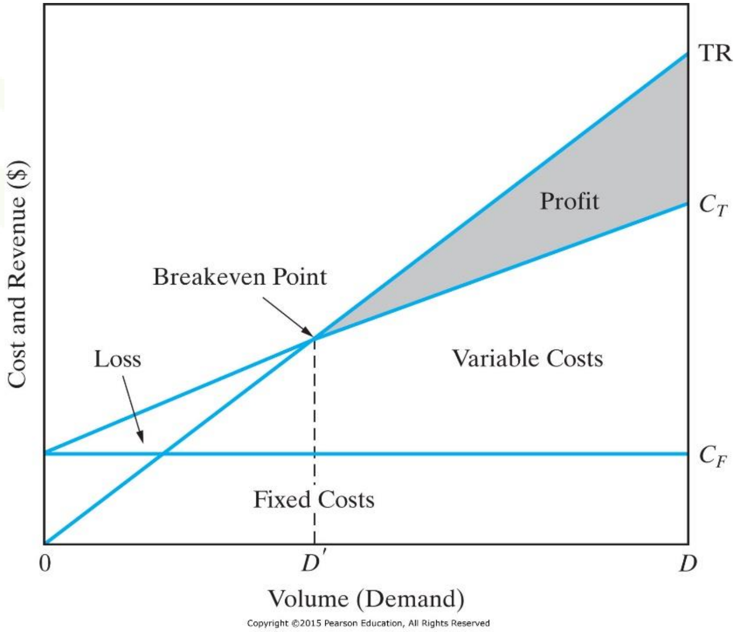
Cost, volume, and breakeven point relationships
Scenario 2: price independent of demand

- $TR = p \cdot D$
- $C_T = C_F + c_v \cdot D$



Example 2-5 Breakeven point when price is indep. of demand

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An engineering consulting firm measures its output in a standard service hour unit, which is a function of the personnel grade levels in the professional staff.

the variable cost (c_v) is \$62 per std service hour

The charge-out rate (selling price) $p = \$85.56$ per hour

The max output of the firm is 160,000 per year

The fixed cost (C_F) is \$2,024,000 per year

a) What is the breakeven point in std hrs and in % total capacity?

b) What is the % reduction in the breakeven point (sensitivity) if fixed costs are reduced 10%; if variable cost per hour is reduced 10%; and if the selling price per unit is increased by 10%?

Example 2-5 Breakeven point when price is indep. of demand

the variable cost (c_v) is \$62 per std service hour
The charge-out rate (selling price) $p = \$85.56$ per hour
The max output of the firm is 160,000 per year
The fixed cost (C_F) is \$2,024,000 per year

a) What is the breakeven point in std hrs and in % total capacity?

$$TR = TC \quad (breakeven\ point)$$

$$pD' = C_F + c_v D'$$

$$D' = \frac{C_F}{(p - c_v)'}$$

$$D' = \frac{\$2.024{,}000}{(\$85.56 - \$62)} = 85{,}908\ h/y$$

$$D'\% = \frac{85{,}908}{160{,}000} = 53.7\%\ of\ capacity$$

Example 2-5 Breakeven point when price is indep. of demand

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the variable cost (c_v) is $62 per std service hour
The charge-out rate (selling price) $p = \$85.56$ per hour
The fixed cost (C_F) is $2,024,000 per year

b) What is the % reduction in the breakeven point (sensitivity) if:

- fixed costs (CF) are reduced 10%;
- variable cost per hour (cv) is reduced 10%;
- the selling price per unit (p) is increased by 10%?

10% reduction in C_F:

$$D' = \frac{90\% \cdot C_F}{(p - c_v)'} = \frac{90\%(\$2.024{,}000)}{(\$85.56 - \$62)} = 77{,}318\ h/y$$

Change in D’: $\frac{85{,}908-77{,}318}{85{,}908} = 10\%$

10% reduction in c_v:

$$D' = \frac{C_F}{(p - 90\% c_v)'} = \frac{\$2.024{,}000}{(\$85.56 - 90\% \cdot \$62)} = 68{,}011\ h/y$$

Change in D’ $\frac{85{,}908-68{,}011}{85{,}908} = 20.8\%$

Example 2-5 Breakeven point when price is indep. of demand

Co-funded by the Erasmus+ Programme of the European Union

the variable cost (c_v) is \$62 per std service hour
The charge-out rate (selling price) $p = \$85.56$ per hour
The fixed cost (C_F) is \$2,024,000 per year

b) What is the % reduction in the breakeven point (sensitivity) if:

- the selling price per unit (p) is increased by 10%?

10% increase in p:

$$D' = \frac{C_F}{(1.1p - c_v)'} = \frac{(\$2.024{,}000)}{(1.1 \cdot \$85.56 - \$62)} = 63{,}021 h/y$$

Change in D': $\frac{85{,}908 - 63{,}021}{85{,}908} = 26.6\%$

2.3 COST DRIVEN OPTIMIZATION

- Engineers must maintain a life-cycle viewpoint
- **Green Engineering:** A movement called *Design for the Environment (DFE),* or "*green engineering*" has prevention of waste, improved materials selection, and reuse and recycling of resources among its goals
- **Cost-effective design**: consider discrete and continuous optimization problems involve a single design variable "*primary cost driver*" X; weight, velocity….
- Tasks for cost-driven design optimization:
  1. Determine optimal value of the design variable
  2. Select the best alternative
- Cost models developed in these problem has three types of costs:
  1. Fixed cost(s), (k)
  2. Direct cost(s) vary with the design variable, (a)
  3. Indirect cost(s) vary with the design variable, (b)

$$Cost = aX + \frac{b}{X} + k$$

2.3 COST DRIVEN OPTIMIZATION

Approach for cost-optimizing a design

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1. Identify the design variable, i.e. primary cost driver
   - Pipe diameter, insulation thickness, reactor residence time, temperature…
2. Write an expression for the cost model in terms of the design variable
3. Set the first derivative of the cost model w.r.t the continuous design variable equals zero.
   - For discrete design variables, compute the value of the cost model for each discrete value over a selected range of potential values
4. Solve the eq. found in (3) for the opt. value of the design variable.
   - For discrete design variables, the opt. value has the min. cost value found in step 3.
5. Use the second derivative of the cost model w.r.t the design variable to determine whether the opt. value corresponds to a global max. or min.

Ex 2-6 How fast should the airplane fly. (continuous)

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The cost of operating a jet-powered commercial (passenger-carrying) airplane varies as the three halves (3/2) power of its velocity; specifically,

$$C_O = knv^{3/2}$$

n: trip length in miles, k: constant of proportionality, v: velocity in miles/h

It is known that at 400 miles/h, the average cost of operation is $300 per mile.

The company that owns the aircraft wants to minimize the cost of operation, but that cost must be balanced against the cost of the passenger's time (C_C), which has been set at $300,000 /h.

a) At what velocity should the trip be planned to minimize the total cost of operating the airplane and the cost of passenger's time>

b) How do you know that your answer for the problem in part a minimizes the total cost?

Ex 2-6 How fast should the airplane fly

$$C_O = knv^{3/2}$$

n: trip length in miles, k: constant of proportionality, v: velocity in miles/h

It is known that at 400 miles/h, the average cost of operation is \$300 per mile.

the passenger's time (C_C) =\$300,000 /h.

a) At what velocity should the trip be planned to minimize the total cost of operating the airplane and the cost of passenger's time?

$$C_T = C_O + C_C = knv^{3/2} + \left(\frac{\$300{,}000}{h}\right)\left(\frac{n}{v}\right)$$

$$\frac{C_O}{n} = \frac{\$300}{mile} = kv^{3/2} = k\left(400\frac{miles}{h}\right)^{3/2}$$

$$k = \$0.0375\left(\frac{h}{mile}\right)^{3/2}$$

$$C_T = \left(0.0375\left(\frac{h}{mile}\right)^{\frac{3}{2}}\right)(n\ miles)\left(v\frac{mile}{h}\right)^{3/2} + \left(\frac{\$300{,}000}{h}\right)\left(\frac{n\ miles}{v\frac{miles}{h}}\right)$$

$$C_T = 0.0375\ nv^{3/2} + \$300{,}000\left(\frac{n}{v}\right)$$

Ex 2-6 How fast should the airplane fly

a) At what velocity should the trip be planned to minimize the total cost of operating the airplane and the cost of passenger's time?

$$C_T = 0.0375\, nv^{3/2} + \$300{,}000\left(\frac{n}{v}\right)$$

First derivative:

$$\frac{dC_T}{dv} = \frac{3}{2}(\$0.0375)nv^{1/2} - \frac{\$300{,}000n}{v^2} = 0$$

$$0.05625v^{1/2} - \frac{300{,}000}{v^2} = 0$$

$$0.05625v^{5/2} - 300{,}000 = 0$$

$$v^{5/2} = 5{,}333{,}333$$

$$v = 491\, mph$$

2.3 COST DRIVEN OPTIMIZATION

Ex 2-6 How fast should the airplane fly

b) How do you know that your answer for the problem in part a minimizes the total cost?

Check the second derivative to confirm a min. cost solution

$$\frac{d^2C_T}{dv^2} = \frac{0.028125}{v^{\frac{1}{2}}} + \frac{600{,}000}{v^3} \qquad , for\ v > 0, and\ therefore, \frac{d^2C_T}{dv^2} > 0$$

The company concludes that v=491 mph minimizes the total cost of this airplane's flight.

END OF PART 2

“The correct solution to any problem depends primarily on a true understanding of what the problem really is”

Arthur Wellington 1987

GJU
الجامعة الألمانية الأردنية
German Jordanian University

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VTC Training

Part 3: Time Value of Money

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1.1 INTRODUCTION

- The objective is to
  - explain the time value of money calculations
  - Illustrate economic equivalence
- **Capital**: wealth in the form of money or property that can be used to produce more wealth
- Engineering economy studies involve the commitment of capital for extended periods of time.
- A dollar today is worth more than a dollar one or more years from now (for several reasons) due to:
  - Interest rate
  - Inflation (التضخم) or deflation (الانكماش)
  - Currency exchange rate
- The riskiest thing a person can do with money is nothing!

4.1 INTRODUCTION

Why consider return to capital?

Why return to capital in the form of interest and profit in the form of profit and interest?

1. pay the providers for its use
2. payments for the risk

Any project or venture must provide a sufficient return to be financially attractive to the suppliers of money or property.

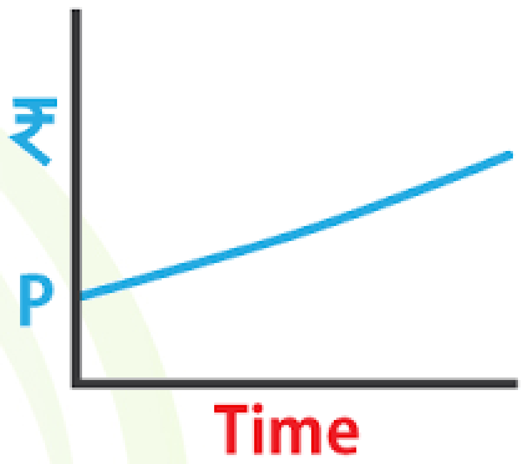
Usury الربا

- Interest existed in Babylon in 2000 B.C
- Typical recorded annual interest rates on loans 6-25 %
- Usury (الربا): high interest rates … prohibited in the Bible and Quran!

4.2 SIMPLE INTEREST الفائدة البسيطة

- When total interest (I) is linearly proportional to:
  - **P**: principal amount lent or borrowed
  - **N**: number of interest periods (e.g., years)
  - *i*: interest rate per interest period

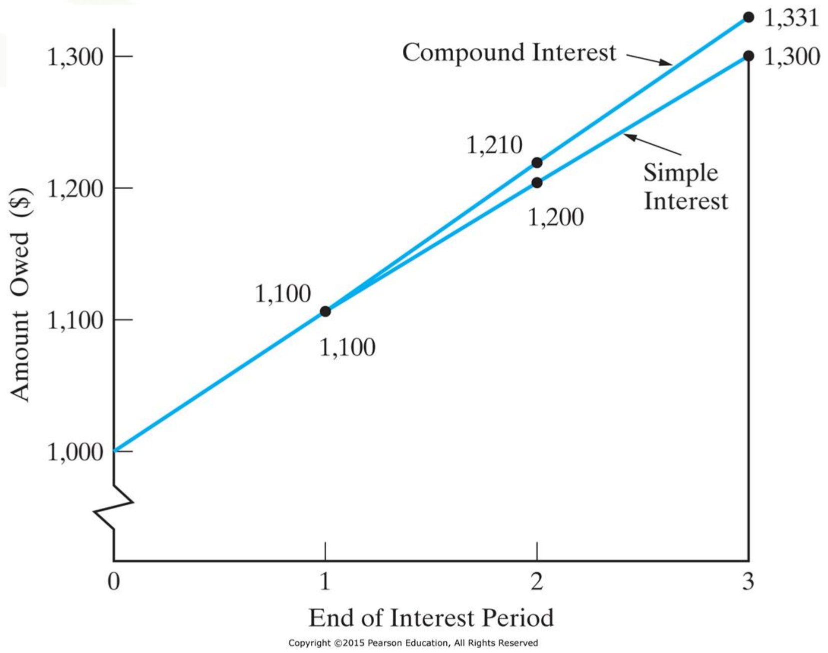
$$I = P \cdot N \cdot i$$



- Not used frequently in modern commercial practice
- The total amount paid at the end of N interest periods = P+I
- Example:
  - Loan: $1,000
  - Period: 3 years
  - Simple interest rate: 10% / y
  - $I = P \cdot N \cdot i = \$1000 \cdot 3 \cdot 10\% = \$300$
  - **Total amount owed = P +I = $1000 + $300 = $1,300**

4.2 COMPOUND INTEREST الفائدة المركبة

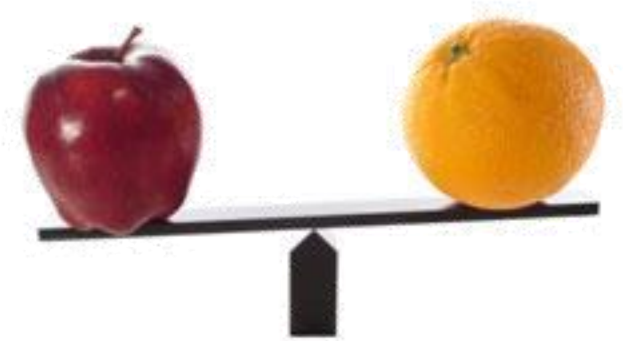
- When the interest charge for any period is based on the remaining principal amount plus any accumulated interest charges up to the beginning of that period
- Example
  - Loan: $1000
  - Periods: 3
  - Interest rate: 10% compounded each period



Period	(1) Amount owed at beginning of period	(2)=(1)*10% interest amount for period	(3)=(1)+(2) amount owed at end of period
1	$1,000	$100	$1,100
2	$1,100	$110	$1,210
3	$1,210	$121	$1,331

4.4 CONCEPT OF EQUIVALENCE

- How can alternatives be compared when interest is involved over extended period of time?
- Compare alternatives by reducing them to an equivalent basis dependent on:
  1. Interest rate
  2. Amount of money involved
  3. Timing of the monetary receipts or expenses
- Using these elements we can “move” cash flows so that we can compare them at particular points in time.



4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES

- $i = effective\ interest\ rate\ per\ interest\ period$
- $N = number\ of\ compounding\ (interest\ period)$
- $P = present\ sum\ of\ money$
- $F = future\ sum\ of\ money$
- $A = end - of - period\ cash\ flows$
- Cash inflows = receipts
- Cash outflows = expenditures
- Net cash flow = Cash inflows - Cash outflows
- The use of cash flow diagrams is strongly recommended

4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES

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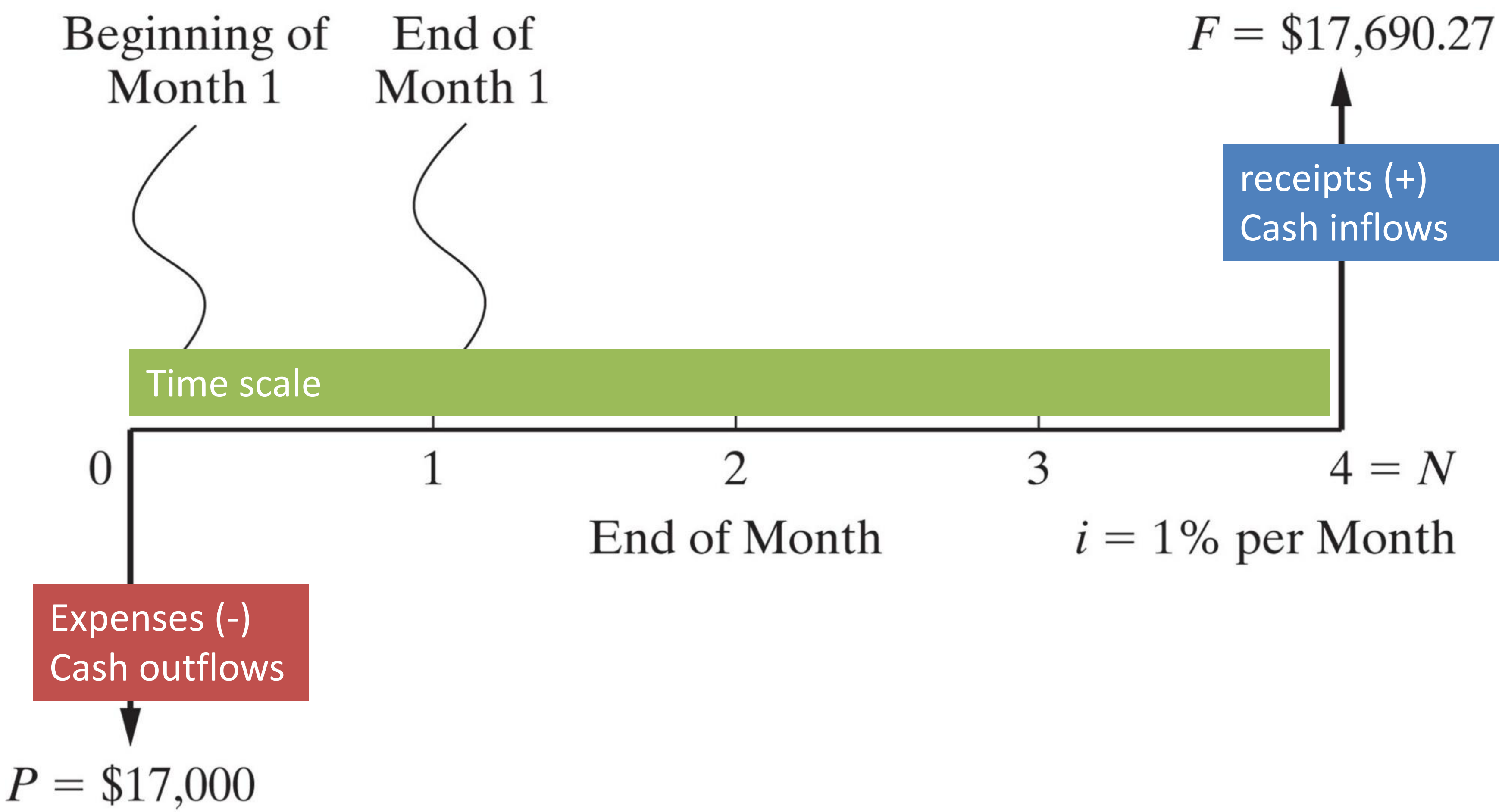
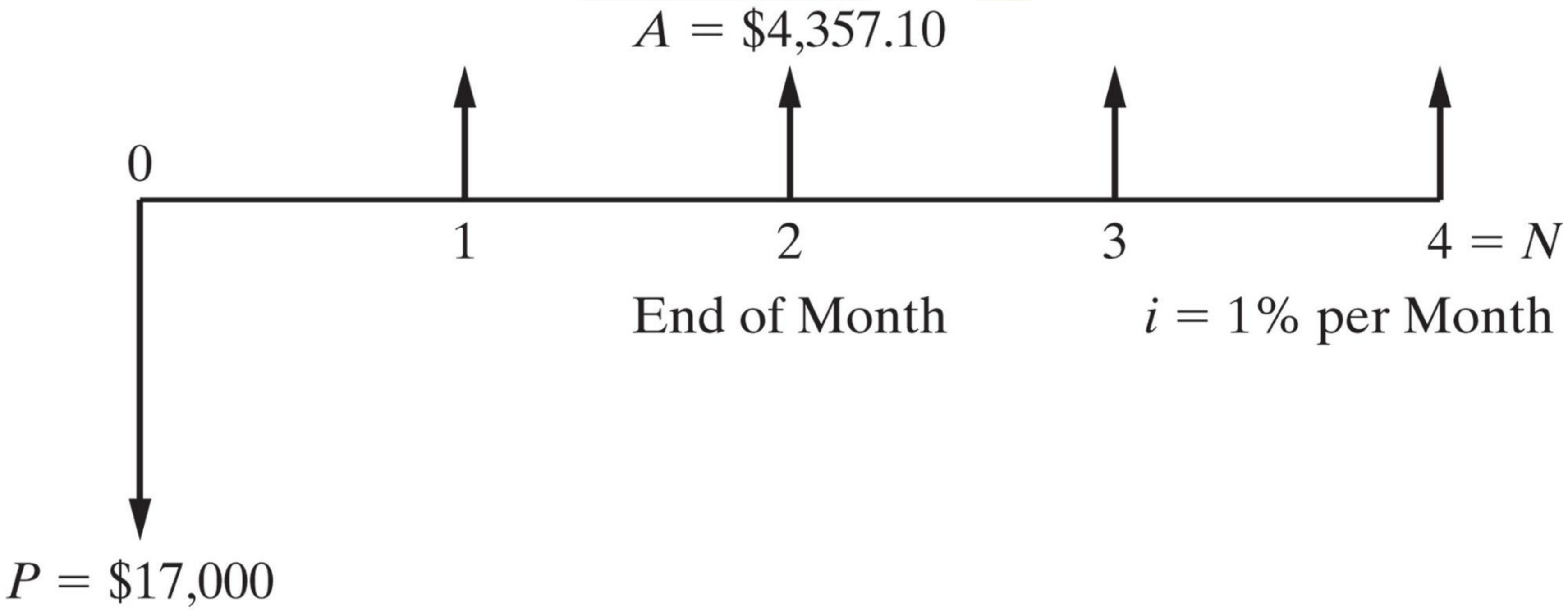


Cash-Flow Diagram for Plan 3 of Table 4-1 (Credit Card Company's Viewpoint)

4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES



Cash-Flow Diagram for Plan 2 of Table 4-1 (Lender's Viewpoint)

4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES

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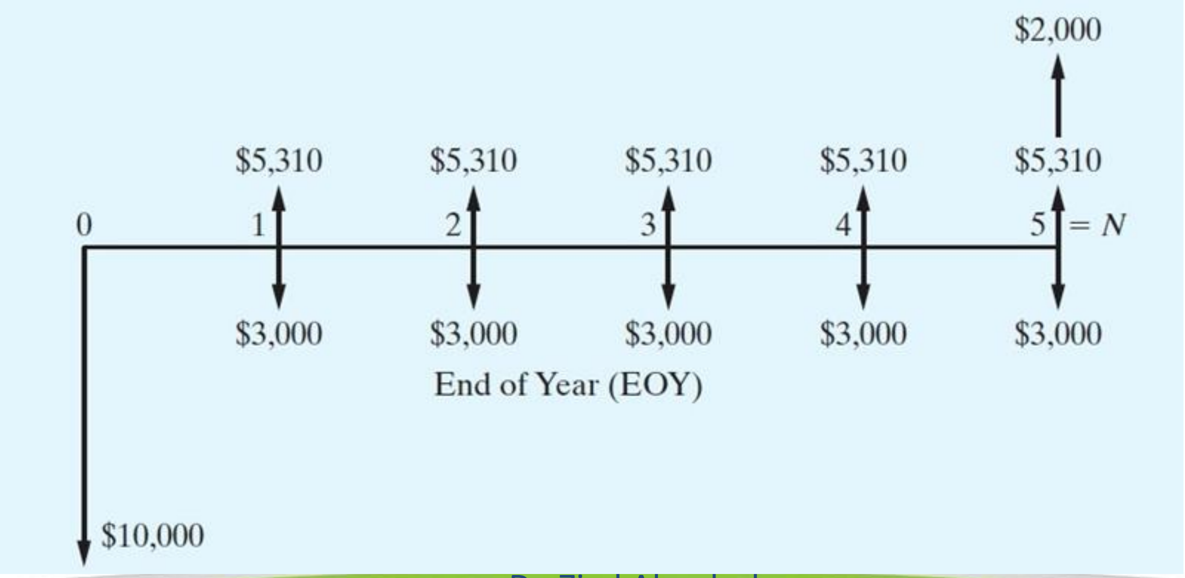
Example 4-1 Cash flow diagramming

Before evaluating the economic merits of a proposed investment, the XYZ corporation insists that its engineers develop a cash flow diagram of the proposal.

An investment of $10,000 can be made that will produce uniform annual revenue of $5,310 for five years and then have a market (recovery) value of $2,000 at the end of year (EOY) five. Annual expenses will be $3,000 at the end of each year for operating and maintaining the project. Draw a cash-flow diagram for the five year life of the project. Use the corporation's viewpoint.

Solution



4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES

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EXAMPLE 4-2 Developing a Net Cash-Flow Table

In a company's renovation of a small office building, two feasible alternatives for upgrading the heating, ventilation, and air conditioning (HVAC) system have been identified. Either Alternative *A* *or* Alternative *B* must be implemented. The costs are as follows:

Alternative *A*	*Rebuild (overhaul) the existing HVAC system*	
	• Equipment, labor, and materials to rebuild	$18,000
	• Annual cost of electricity	32,000
	• Annual maintenance expenses	2,400
Alternative *B*	*Install a new HVAC system that utilizes existing ductwork*	
	• Equipment, labor, and materials to install	$60,000
	• Annual cost of electricity	9,000
	• Annual maintenance expenses	16,000
	• Replacement of a major component four years hence	9,400

At the end of eight years, the estimated market value for Alternative *A* is $2,000 and for Alternative *B* it is $8,000. Assume that both alternatives will provide comparable service (comfort) over an eight-year period, and assume that the major component replaced in Alternative *B* will have no market value at EOY eight. (1) Use a cash-flow table and end-of-year convention to tabulate the net cash flows for both alternatives. (2) Determine the annual net cash-flow difference between the alternatives ($B - A$).

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= – 25000 – 9400

= C3 – B3

= SUM(D$3:D3)

	A	B	C	D	E
1		Alternative *A*	Alternative *B*	Difference	Cumulative
2	End of Year	Net Cash Flow	Net Cash Flow	(*B*-*A*)	Difference
3	0 (now)	$ (18,000)	$ (60,000)	$ (42,000)	$ (42,000)
4	1	$ (34,400)	$ (25,000)	$ 9,400	$ (32,600)
5	2	$ (34,400)	$ (25,000)	$ 9,400	$ (23,200)
6	3	$ (34,400)	$ (25,000)	$ 9,400	$ (13,800)
7	4	$ (34,400)	$ (34,400)	$ -	$ (13,800)
8	5	$ (34,400)	$ (25,000)	$ 9,400	$ (4,400)
9	6	$ (34,400)	$ (25,000)	$ 9,400	$ 5,000
10	7	$ (34,400)	$ (25,000)	$ 9,400	$ 14,400
11	8	$ (32,400)	$ (17,000)	$ 15,400	$ 29,800
12	Total	$ (291,200)	$ (261,400)		

= – 34400 + 2000

= – 25000 + 8000

= SUM(B3:B11)

4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES

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EOY	Alternative A Net Cash Flow	Alternative B Net Cash Flow	Difference (B-A)	Cumulative Difference
0	$ (18,000)	$ (60,000)	$ (42,000)	$ (42,000)
1	$ (34,400)	$ (25,000)	$ 9,400	$ (32,600)
2	$ (34,400)	$ (25,000)	$ 9,400	$ (23,200)
3	$ (34,400)	$ (25,000)	$ 9,400	$ (13,800)
4	$ (34,400)	$ (34,400)	$ -	$ (13,800)
5	$ (34,400)	$ (25,000)	$ 9,400	$ (4,400)
6	$ (34,400)	$ (25,000)	$ 9,400	$ 5,000
7	$ (34,400)	$ (25,000)	$ 9,400	$ 14,400
8	$ (32,400)	$ (17,000)	$ 15,400	$ 29,800
Total	**$ (291,200)**	**$ (261,400)**		
EoY	End of year			

4.5 NOTATION AND CASH-FLOW DIAGRAMS AND TABLES

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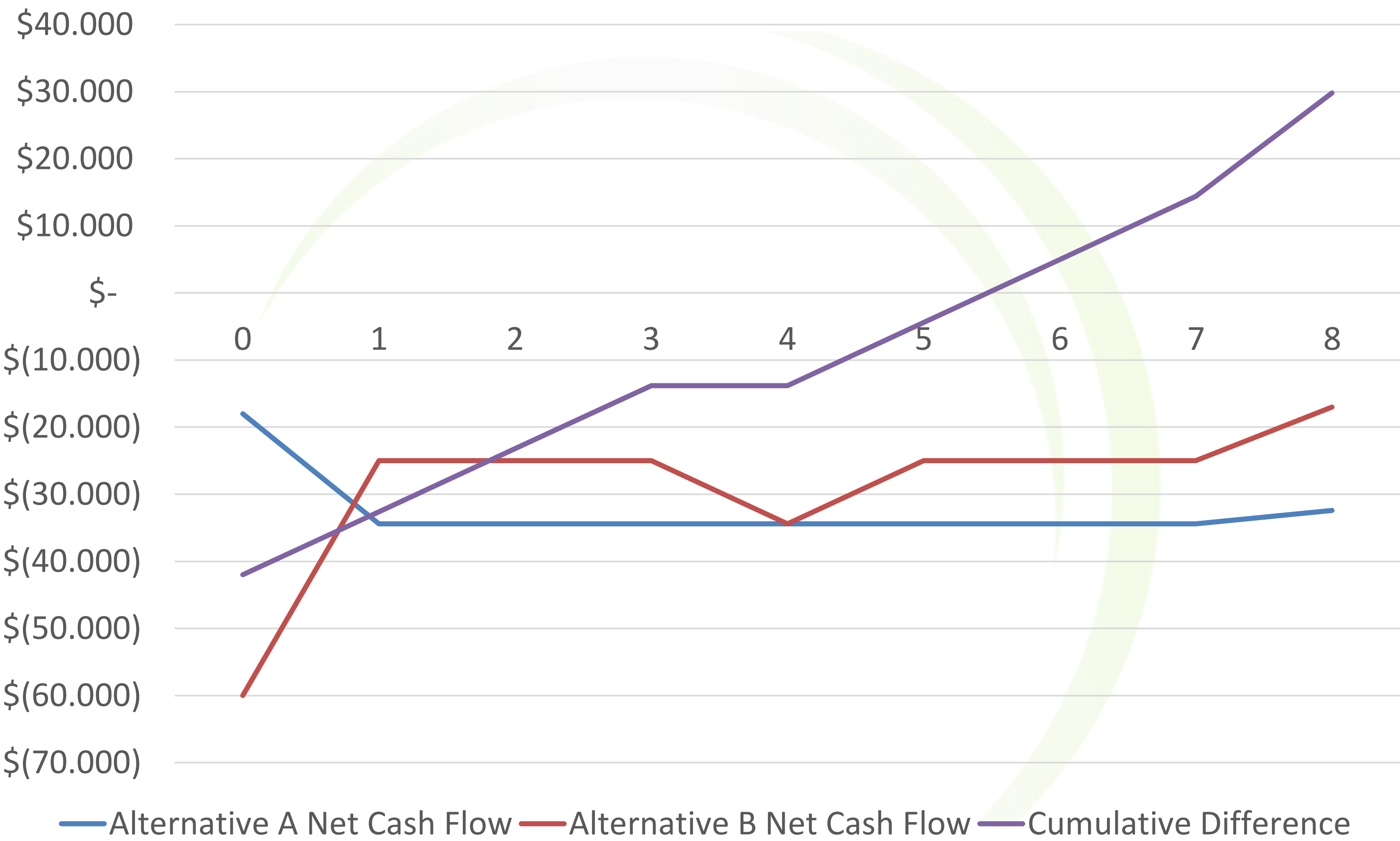

4.6 RELATING PRESENT & FUTURE EQUIVALENT VALUES OF SINGLE CASH FLOWS

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EXAMPLE 4-3 **Future Equivalent of a Present Sum**

Suppose that you borrow \$8,000 now, promising to repay the loan principal plus accumulated interest in four years at $i = 10\%$ per year. How much would you repay at the end of four years?

Solution

Year	Amount Owed at Start of Year	Interest Owed for Each Year	Amount Owed at End of Year	Total End-of-Year Payment
1	$P = \$\ 8{,}000$	$iP = \$\ 800$	$P(1+i) = \$\ 8{,}800$	0
2	$P(1+i) = \$\ 8{,}800$	$iP(1+i) = \$\ 880$	$P(1+i)^2 = \$\ 9{,}680$	0
3	$P(1+i)^2 = \$\ 9{,}680$	$iP(1+i)^2 = \$\ 968$	$P(1+i)^3 = \$10{,}648$	0
4	$P(1+i)^3 = \$10{,}648$	$iP(1+i)^3 = \$1{,}065$	$P(1+i)^4 = \$11{,}713$	$F = \$11{,}713$

In general, we see that $F = P(1+i)^N$, and the total amount to be repaid is \$11,713.

4.6 RELATING PRESENT & FUTURE EQUIVALENT VALUES OF SINGLE CASH FLOWS

Finding P when Given F

$$P = F(1 + i)^{-N}$$

$P = present\ sum\ of\ money$

$F = future\ sum\ of\ money$

$i = effective\ interest\ rate\ per\ interest\ period$

$N = number\ of$ periods

$(1 + i)^{N} = single\ payment\ present\ worth\ factor$

Functional Symbols: Use App C

$F = F(F/P, i\%, N)$: Find F given P at i% interest per period for N interest periods

$P = F(p/F, i\%, N)$: Find P given F at i% interest per period for N interest periods

4.6 RELATING PRESENT & FUTURE EQUIVALENT VALUES OF SINGLE CASH FLOWS

Finding i when Given P, F, N

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$$i = \sqrt[N]{F/P} - 1$$

$$i = (F/P)^{-N} - 1$$

$P = present\ sum\ of\ money$

$F = future\ sum\ of\ money$

$i = effective\ interest\ rate\ per\ interest\ period$

$N = number\ of$ periods

EXAMPLE 4-5 **The Inflating Price of Gasoline**

VIDEO SOLUTIONS

The average price of gasoline in 2005 was \$2.31 per gallon. In 1993, the average price was \$1.07.* What was the average annual rate of increase in the price of gasoline over this 12-year period?

Solution

With respect to the year 1993, the year 2005 is in the future. Thus, $P = \$1.07$, $F = \$2.31$, and $N = 12$. Using Equation (4-6), we find $i = \sqrt[12]{2.31/1.07} - 1 = 0.0662$ or 6.62% per year.

* This data was obtained from the Energy Information Administration of the Department of Energy. Historical prices of gasoline and other energy sources can be found at www.eia.doe.gov.

4.6 RELATING PRESENT & FUTURE EQUIVALENT VALUES OF SINGLE CASH FLOWS

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Finding N when Given P, F, i

$$F = P(1+i)^N$$

$$(1+i)^N = (F/P)$$

$$N\log(1+i) = \log(F/P)$$

$$N = \frac{\log(F/P)}{\log(1+i)}$$

EXAMPLE 4-6 **When Will Gasoline Cost \$5.00 per Gallon?**

In Example 4-5, the average price of gasoline was given as \$2.31 in 2005. We computed the average annual rate of increase in the price of gasoline to be 6.62%. If we assume that the price of gasoline will continue to inflate at this rate, how long will it be before we are paying \$5.00 per gallon?

Solution

We have $P = \$2.31$, $F = \$5.00$, and $i = 6.62\%$ per year. Using Equation (4-7), we find

$$N = \frac{\log(\$5.00/\$2.31)}{\log(1+0.0662)} = \frac{\log(2.1645)}{\log(1.0662)} = 12.05 \text{ years.}$$

So, if gasoline prices continue to increase at the same rate, we can expect to be paying \$5.00 per gallon in 2017.

Finding F when given A

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]$$

$A = \textit{Annuity occurs at the end of period 1 through N, inclusive}$

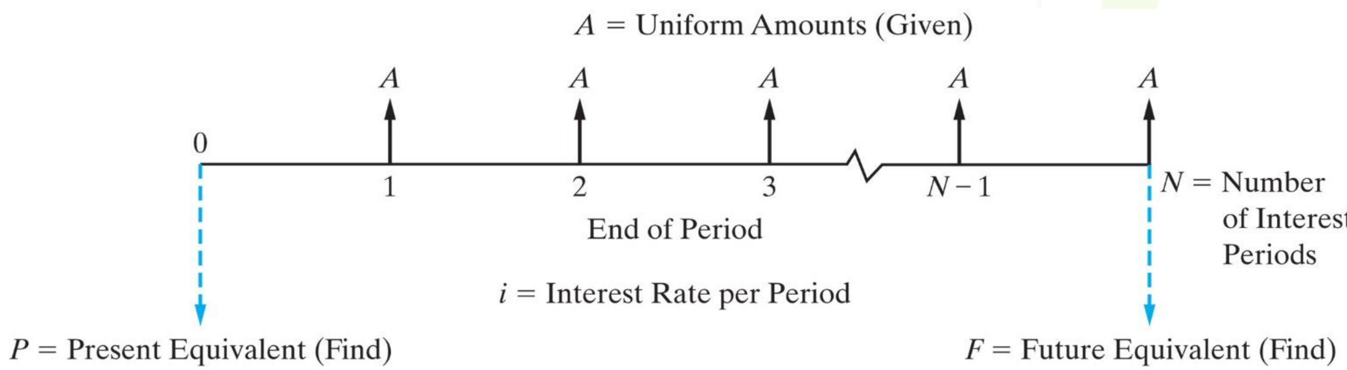
$\left[\frac{(1+i)^N - 1}{i}\right] = \textit{uniform series compound amount factor}$

$P = \textit{present value occurs one interest period before first } A$

$F = \textit{Future value occurs at the same time as the last } A, \textit{and } N \textit{ periods after } P$

$i = \textit{effective interest rate per interest period}$

$N = \textit{number of}$ periods



Cash-Flow Diagram Relating Uniform Series to Its P and F

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

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EXAMPLE 4-7 **Future Value of a College Degree**

A recent government study reported that a college degree is worth an extra \$23,000 per year in income (A) compared to what a high-school graduate makes. If the interest rate (i) is 6% per year and you work for 40 years (N), what is the future compound amount (F) of this extra income?

Solution

The viewpoint we will use to solve this problem is that of "lending" the \$23,000 of extra annual income to a savings account (or some other investment vehicle). The future equivalent is the amount that can be withdrawn after the 40th deposit is made.

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

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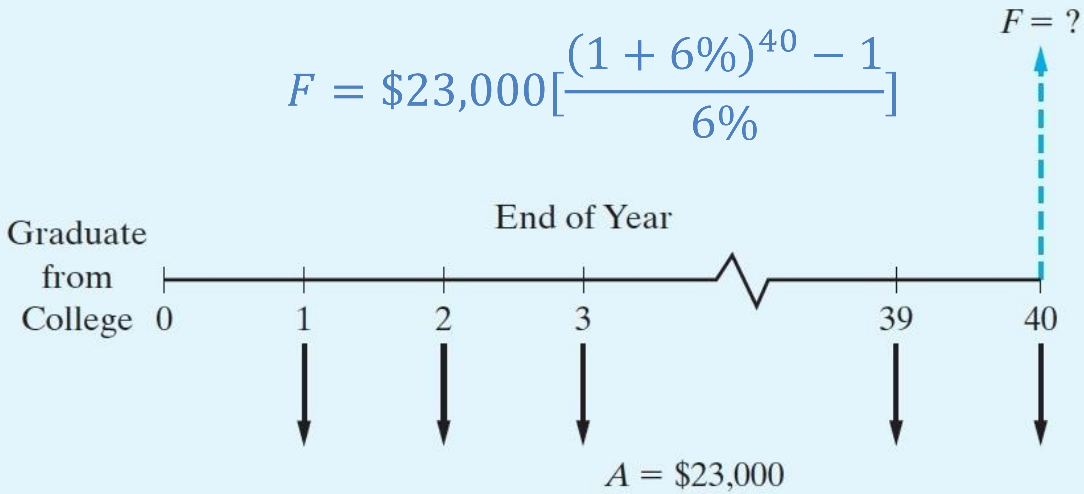
EXAMPLE 4-7 Future Value of a College Degree

$$F = \$23{,}000\left[\frac{(1+6\%)^{40}-1}{6\%}\right]$$



Notice that the future equivalent occurs at the *same time* as the last deposit of \$23,000.

$$\begin{aligned} F &= \$23{,}000(F/A,\ 6\%,\ 40) \\ &= \$23{,}000(154.762) \\ &= \$3{,}559{,}526 \end{aligned}$$

The bottom line is "Get your college degree!"

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

Finding P when given A

$$F = A[\frac{(1+i)^N - 1}{i}]$$

$$F = P(1+i)^N$$

$$P(1+i)^N = A[\frac{(1+i)^N - 1}{i}]$$

$$P = A[\frac{(1+i)^N - 1}{i(1+i)^N}]$$

Finding P when given A

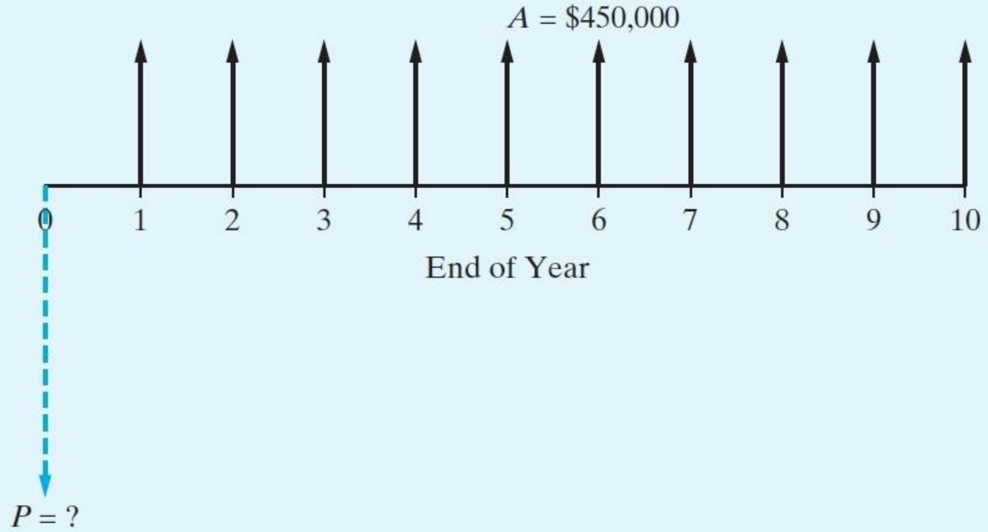
EXAMPLE 4-9 **Present Equivalent of an Annuity (Uniform Series)**

VIDEO SOLUTIONS

A micro-brewery is considering the installation of a newly designed boiler system that burns the dried, spent malt and barley grains from the brewing process. The boiler will produce process steam that powers the majority of the brewery's energy operations, saving \$450,000 per year over the boiler's expected life of 10 years. If the interest rate is 12% per year, how much money can the brewery afford to invest in the new boiler system?

Solution

In the cash flow diagram below, notice that the affordable amount (i.e., the present equivalent, P) occurs one time period (year) before the first end-of-year cash flow of \$450,000.



4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

Finding P when given A

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

EXAMPLE 4-9 **Present Equivalent of an Annuity (Uniform Series)**

The increase in annual cash flow is \$450,000, and it continues for 10 years at 12% annual interest. The upper limit on what the brewery can afford to spend on the new boiler is:

$$P = \$450{,}000\ (P/A, 12\%, 10)$$

$$= \$450{,}000\ (5.6502)$$

$$= \$2{,}542{,}590.$$

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

Finding P when given A

EXAMPLE 4-10 How Much Is a Lifetime Oil Change Offer Worth?

"Make your best deal with us on a new automobile and we'll change your oil for free for as long as you own the car!" If you purchase a car from this dealership, you expect to have four free oil changes per year during the five years you keep the car. Each oil change would normally cost you \$30. If you save your money in a mutual fund earning 2% per quarter, how much are the oil changes worth to you at the time you buy the car?

Solution

In this example, we need to find the present equivalent of the cost of future oil changes. The cash-flow diagram is shown below. Notice that P occurs one time period (a quarter of a year, in this example) before the first oil change cash flow (A).

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

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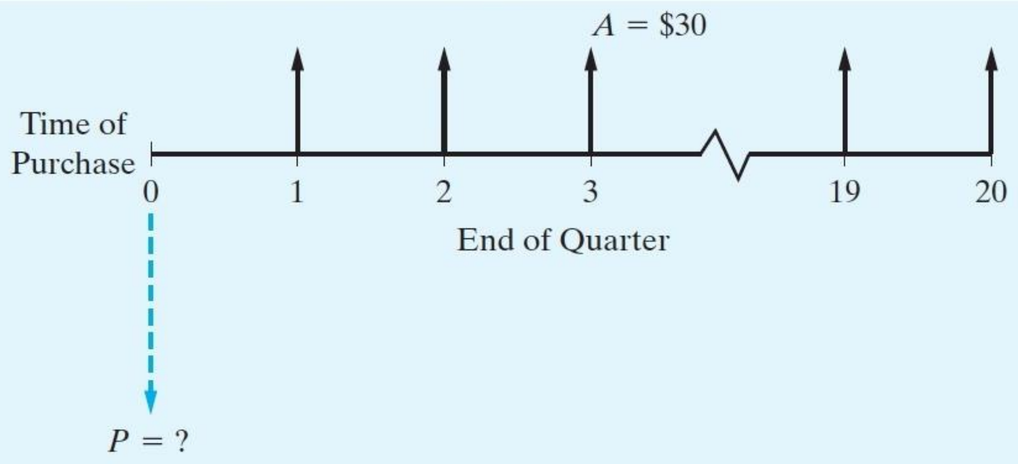
Finding P when given A

EXAMPLE 4-10 **How Much Is a Lifetime Oil Change Offer Worth?**



The interest rate is 2% per quarter, and a total of (4 oil changes/year × 5 years) = 20 oil changes (cash flows) are anticipated.

$$P = \$30(P/A, 2\%, 20)$$
$$= \$30(16.3514)$$
$$= \$490.54$$

Now you are in a position to determine how great of a deal you are being offered. If the best price of another dealership is more than \$490.54 cheaper than what you are being offered at this dealership, maybe this deal isn't so great.

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

Finding A when given F or P

$$A = F\left[\frac{i}{(1+i)^N - 1}\right]$$

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

Finding A when given F or P

EXAMPLE 4-11 **Computing Your Monthly Car Payment**

VIDEO SOLUTIONS

You borrow \$15,000 from your credit union to purchase a used car. The interest rate on your loan is 0.25% per month* and you will make a total of 36 monthly payments. What is your monthly payment?

* A good credit score (rating) can help you secure lower interest rates on loans. The Web site www.annualcreditreport.com allows you to check your credit score once per year at no cost.

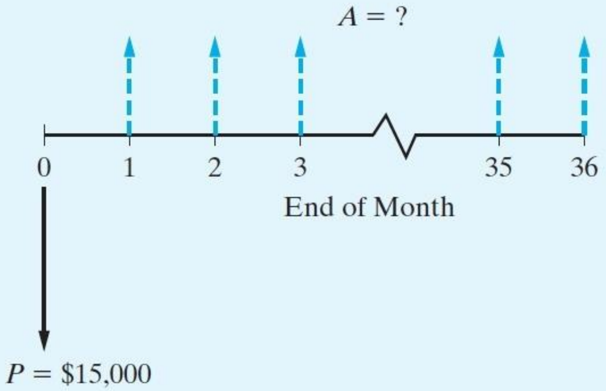
Finding A when given F or P

EXAMPLE 4-11 Computing Your Monthly Car Payment

Solution

The cash-flow diagram shown below is drawn from the viewpoint of the bank. Notice that the present amount of $15,000 occurs one month (interest period) *before* the first cash flow of the uniform repayment series.

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$



The amount of the car payment is easily calculated using Equation (4-15).

$$A = \$15{,}000(A/P, {}^1\!/\!_4\%, 36)$$
$$= \$15{,}000(0.0291)$$
$$= \$436.50 \text{ per month}$$

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Finding N given A, P, i

EXAMPLE 4-12 **Prepaying a Loan—Finding *N***

Your company has a \$100,000 loan for a new security system it just bought. The annual payment is \$8,880 and the interest rate is 8% per year for 30 years. Your company decides that it can afford to pay \$10,000 per year. After how many payments (years) will the loan be paid off?

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$

Solution

The original loan payment was found using Equation (4-15).

$$A = \$100{,}000\ (A/P, 8\%, 30) = \$100{,}000\ (0.0888) = \$8{,}800 \text{ per year}$$

Now, instead of paying \$8,880 per year, your company is going to pay \$10,000 per year. Common sense tells us that less than 30 payments will be necessary to pay off the \$100,000 loan. Using Equation (4-11), we find

$$\$100{,}000 = \$10{,}000\ (P/A, 8\%, N)$$

$$(P/A, 8\%, N) = 10.$$

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Finding N given A, P, i

EXAMPLE 4-12 **Prepaying a Loan—Finding *N***

We can now use the interest tables provided in Appendix C to find N. Looking down the Present Worth Factor column (P/A) of Table C-11, we see that

$$(P/A,\ 8\%,\ 20) = 9.8181$$

and

$$(P/A,\ 8\%,\ 21) = 10.0168.$$

So, if \$10,000 is paid per year, the loan will be paid off after 21 years instead of 30. The exact amount of the 21st payment will be slightly less than \$10,000 (but we'll save that solution for another example).

4.7 RELATING A UNIFORM SERIES (ANNUITY) TO ITS PRESENT AND FUTURE VALUES

Finding N given A, P, i

TABLE C-11 Discrete Compounding; $i = 8\%$

	Single Payment		Uniform Series				Uniform Gradient	
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor
N	To Find *F* Given *P* *F/P*	To Find *P* Given *F* *P/F*	To Find *F* Given *A* *F/A*	To Find *P* Given *A* *P/A*	To Find *A* Given *F* *A/F*	To Find *A* Given *P* *A/P*	To Find *P* Given *G* *P/G*	To Find *A* Given *G* *A/G*
1	1.0800	0.9259	1.0000	0.9259	1.0000	1.0800	0.000	0.0000
2	1.1664	0.8573	2.0800	1.7833	0.4808	0.5608	0.857	0.4808
3	1.2597	0.7938	3.2464	2.5771	0.3080	0.3880	2.445	0.9487
4	1.3605	0.7350	4.5061	3.3121	0.2219	0.3019	4.650	1.4040
5	1.4693	0.6806	5.8666	3.9927	0.1705	0.2505	7.372	1.8465
6	1.5869	0.6302	7.3359	4.6229	0.1363	0.2163	10.523	2.2763
7	1.7138	0.5835	8.9228	5.2064	0.1121	0.1921	14.024	2.6937
8	1.8509	0.5403	10.6366	5.7466	0.0940	0.1740	17.806	3.0985
9	1.9990	0.5002	12.4876	6.2469	0.0801	0.1601	21.808	3.4910
10	2.1589	0.4632	14.4866	6.7101	0.0690	0.1490	25.977	3.8713
11	2.3316	0.4289	16.6455	7.1390	0.0601	0.1401	30.266	4.2395
12	2.5182	0.3971	18.9771	7.5361	0.0527	0.1327	34.634	4.5957
13	2.7196	0.3677	21.4953	7.9038	0.0465	0.1265	39.046	4.9402
14	2.9372	0.3405	24.2149	8.2442	0.0413	0.1213	43.472	5.2731
15	3.1722	0.3152	27.1521	8.5595	0.0368	0.1168	47.886	5.5945
16	3.4259	0.2919	30.3243	8.8514	0.0330	0.1130	52.264	5.9046
17	3.7000	0.2703	33.7502	9.1216	0.0296	0.1096	56.588	6.2037
18	3.9960	0.2502	37.4502	9.3719	0.0267	0.1067	60.843	6.4920
19	4.3157	0.2317	41.4463	9.6036	0.0241	0.1041	65.013	6.7697
20	4.6610	0.2145	45.7620	9.8181	0.0219	0.1019	69.090	7.0369
21	5.0338	0.1987	50.4229	10.0168	0.0198	0.0998	73.063	7.2940
22	5.4365	0.1839	55.4568	10.2007	0.0180	0.0980	76.926	7.5412
23	5.8715	0.1703	60.8933	10.3711	0.0164	0.0964	80.673	7.7786
24	6.3412	0.1577	66.7648	10.5288	0.0150	0.0950	84.300	8.0066
25	6.8485	0.1460	73.1059	10.6748	0.0137	0.0937	87.804	8.2254
30	10.0627	0.0994	113.2832	11.2578	0.0088	0.0888	103.456	9.1897
35	14.7853	0.0676	172.3168	11.6546	0.0058	0.0858	116.092	9.9611
40	21.7245	0.0460	259.0565	11.9246	0.0039	0.0839	126.042	10.5699
45	31.9204	0.0313	386.5056	12.1084	0.0026	0.0826	133.733	11.0447
50	46.9016	0.0213	573.7702	12.2335	0.0017	0.0817	139.593	11.4107
60	101.2571	0.0099	1253.2133	12.3766	0.0008	0.0808	147.300	11.9015
80	471.9548	0.0021	5886.9354	12.4735	0.0002	0.0802	153.800	12.3301
100	2199.7613	0.0005	27484.5157	12.4943	a	0.0800	155.611	12.4545
∞				12.5000		0.0800		

a Less than 0.0001.

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Finding N given A, P, i

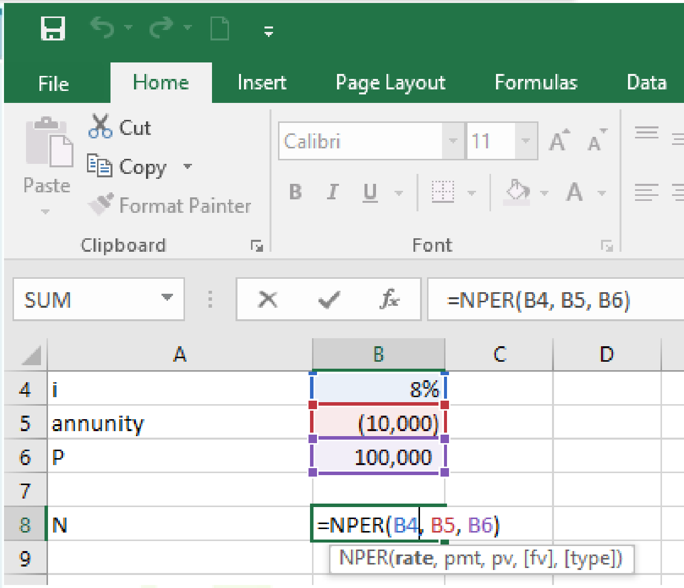
Prepaying a Loan—Finding *N*

Spreadsheet Solution

There is a financial function in Excel that would allow us to solve for the unknown number of periods. NPER(*rate*, *pmt*, *pv*) will compute the number of payments of magnitude *pmt* required to pay off a present amount (*pv*) at a fixed interest rate (*rate*).

$$N = \text{NPER}(0.08, -10000, 100000) = 20.91$$

Note that from your company's viewpoint, it received \$100,000 (a cash inflow) at time 0 and is making \$10,000 payments (cash outflows). Hence the annuity is expressed as a negative number in NPER( ) and the present amount as a positive number. If we were to reverse the signs—which would represent the lender's viewpoint—the same result would be obtained, namely NPER(0.08, 10000, −100000) = 20.91.



Comment

Prepaying loans can save thousands of dollars in interest. For example, look at the total interest paid under these two repayment plans.

Original payment plan (\$8,880 per year for 30 years):

$$\text{Total interest paid} = \$8{,}880 \times 30 - \$100{,}000 = \$166{,}400$$

New payment plan (\$10,000 per year for 21 years):

$$\text{Total interest paid} = \$10{,}000 \times 21 - \$100{,}000 = \$110{,}000$$

Prepaying the loan in this way would save \$56,400 in interest!

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Finding i, given A, F, and N

EXAMPLE 4-13 **Finding the Interest Rate to Meet an Investment Goal**

After years of being a poor, debt-encumbered college student, you decide that you want to pay for your dream car in cash. Not having enough money now, you decide to specifically put money away each year in a "dream car" fund. The car you want to buy will cost \$60,000 in eight years. You are going to put aside \$6,000 each year (for eight years) to save for this. At what interest rate must you invest your money to achieve your goal of having enough to purchase the car after eight years?

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]$$

Solution

We can use Equation (4-9) to show our desired equivalence relationship.

$$\$60{,}000 = \$6{,}000\,(F/A,\ i\%,\ 8)$$

$$(F/A,\ i\%,\ 8) = 10$$

Finding i, given A, F, and N

EXAMPLE 4-13 **Finding the Interest Rate to Meet an Investment Goal**

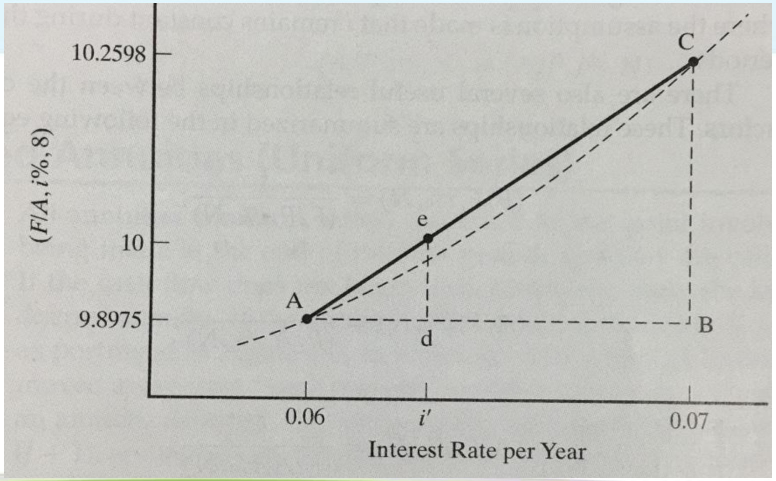
Now we can use the interest tables in Appendix C to help track down the unknown value of i. What we are looking for are two interest rates, one with an $(F/A, i\%, 8)$ value greater than 10 and one with an $(F/A, i\%, 8)$ less than 10. Thumbing through Appendix C, we find

$$(F/A, 6\%, 8) = 9.8975 \quad \text{and} \quad (F/A, 7\%, 8) = 10.2598,$$

which tells us that the interest rate we are looking for is between 6% and 7% per year. Even though the function $(F/A, i\%, N)$ is nonlinear, we can use linear interpolation to approximate the value of i.

The dashed curve in Figure 4-8 is what we are linearly approximating. The answer, i', can be determined by using the similar triangles dashed in Figure 4-8.

$$\frac{\text{line dA}}{\text{line ed}} = \frac{\text{line BA}}{\text{line CB}}$$



Finding i, given A, F, and N

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EXAMPLE 4-13 **Finding the Interest Rate to Meet an Investment Goal**

$$\frac{i' - 0.06}{10 - 9.8975} = \frac{0.07 - 0.06}{10.2598 - 9.8975}$$

$$i' = 0.0628 \text{ or } 6.28\% \text{ per year}$$

So if you can find an investment account that will earn at least 6.28% interest per year, you'll have the $60,000 you need to buy your dream car in eight years.

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Finding i, given A, F, and N

EXAMPLE 4-13 **Finding the Interest Rate to Meet an Investment Goal**

Spreadsheet Solution

Excel has another financial function that allows you to solve for an unknown interest rate. RATE(*nper*, *pmt*, *pv*, *fv*) will return the fixed interest rate that equates an annuity of magnitude *pmt* that lasts for *nper* periods to either its present value (*pv*) or its future value (*fv*).

$$i' = \text{RATE}(8, -6000, 0, 60000) = 0.0629 \text{ or } 6.29\% \text{ per year}$$

Note that a 0 was entered for *pv* since we were working with a known future value in this example.

SUM | =RATE(B1, B2, B

	A	B	C	D	E
1	N	8			
2	A	(6,000)			
3	P	0			
4	F	60,000			
5					
6	i'	=RATE(B1, B2, B3, B4)			
7		RATE(nper, pmt, pv, [fv], [type], [guess])			

i' =6.29%

4.8 SUMMARY OF FORMULAS FOR DISCRETE COMPOUNDING

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TABLE 4-3 Discrete Compounding-Interest Factors and Symbols[a]

To Find:	Given:	Factor by which to Multiply "Given"[a]	Factor Name	Factor Functional Symbol[b]
For single cash flows:				
F	P	$(1+i)^N$	Single payment compound amount	$(F/P, i\%, N)$
P	F	$\frac{1}{(1+i)^N}$	Single payment present worth	$(P/F, i\%, N)$
For uniform series (annuities):				
F	A	$\frac{(1+i)^N-1}{i}$	Uniform series compound amount	$(F/A, i\%, N)$
P	A	$\frac{(1+i)^N-1}{i(1+i)^N}$	Uniform series present worth	$(P/A, i\%, N)$
A	F	$\frac{i}{(1+i)^N-1}$	Sinking fund	$(A/F, i\%, N)$
A	P	$\frac{i(1+i)^N}{(1+i)^N-1}$	Capital recovery	$(A/P, i\%, N)$

[a] i equals effective interest rate per interest period; N, number of interest periods; A, uniform series amount (occurs at the end of each interest period); F, future equivalent; P, present equivalent.

[b] The functional symbol system is used throughout this book.

4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

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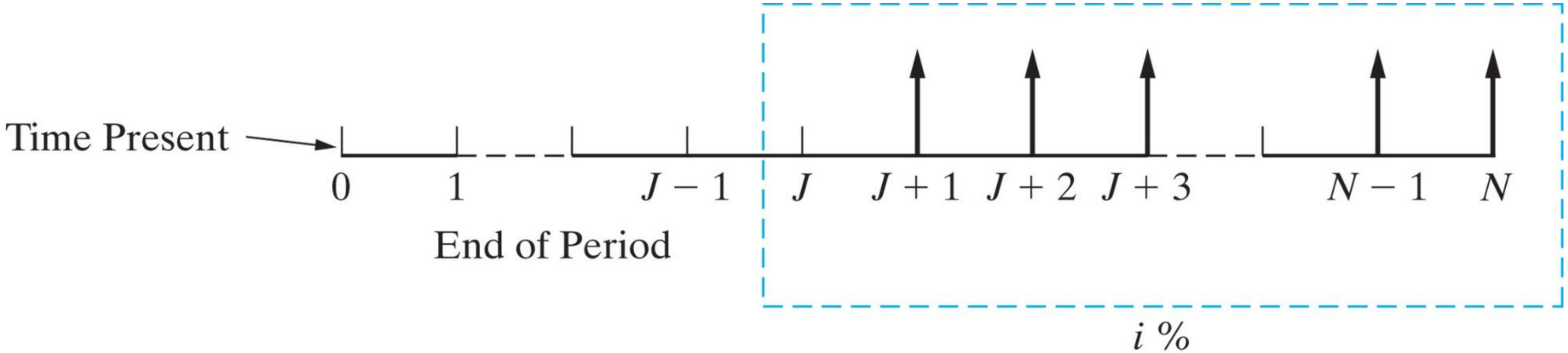
- **Ordinary annuities**: the first cash flow being made at the end of the first period
- **Deferred annuities**: the cash flow does not begin until some later date
- If annuity is deferred for $J\ periods\ (J < N)$
- The first payment is made at the end of $period\ (J + 1)$
- The present equivalent at the end of period J of an annuity with cash flows of amount A is
- $P_o = A(P/A, i\%, N - J)(P/F, i\%, J)$



4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-14

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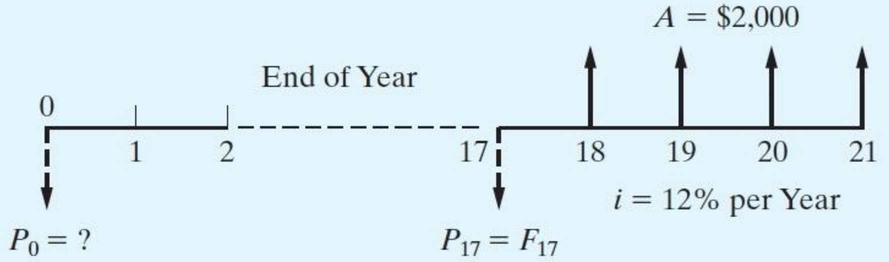
EXAMPLE 4-14 **Present Equivalent of a Deferred Annuity**

VIDEO SOLUTIONS

To illustrate the preceding discussion, suppose that a father, on the day his son is born, wishes to determine what lump amount would have to be paid into an account bearing interest of 12% per year to provide withdrawals of \$2,000 on each of the son's 18th, 19th, 20th, and 21st birthdays.

Hence,

$$P_{17} = A(P/A, 12\%, 4) = \$2{,}000(3.0373) = \$6{,}074.60.$$



Note the dashed arrow in the cash-flow diagram denoting P_{17}. Now that P_{17} is known, the next step is to calculate P_0. With respect to P_0, P_{17} is a future equivalent, and hence it could also be denoted F_{17}. Money at a given point in time, such as the end of period 17, is the same regardless of whether it is called a present equivalent or a future equivalent. Hence,

$$P_0 = F_{17}(P/F, 12\%, 17) = \$6{,}074.60(0.1456) = \$884.46,$$

which is the amount that the father would have to deposit on the day his son is born.

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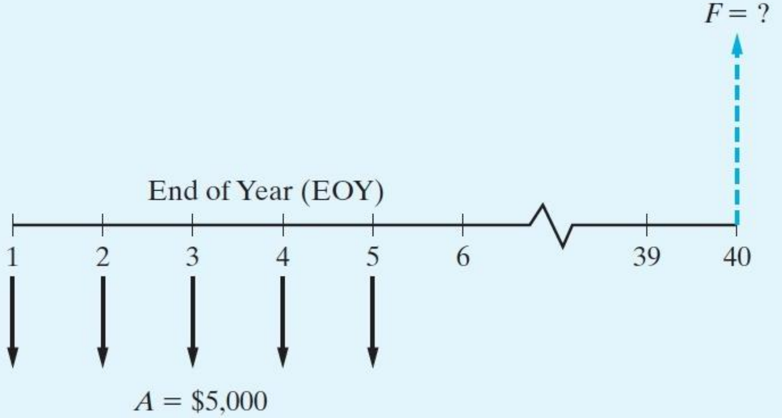
Ex 4-15

EXAMPLE 4-15 **Deferred Future Value of an Annuity**

When you take your first job, you decide to start saving right away for your retirement. You put \$5,000 per year into the company's 401(k) plan, which averages 8% interest per year. Five years later, you move to another job and start a new 401(k) plan. You never get around to merging the funds in the two plans. If the first plan continued to earn interest at the rate of 8% per year for 35 years after you stopped making contributions, how much is the account worth?

Solution

The following cash-flow diagram clarifies the timing of the cash flows for the original 401(k) investment plan.



4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

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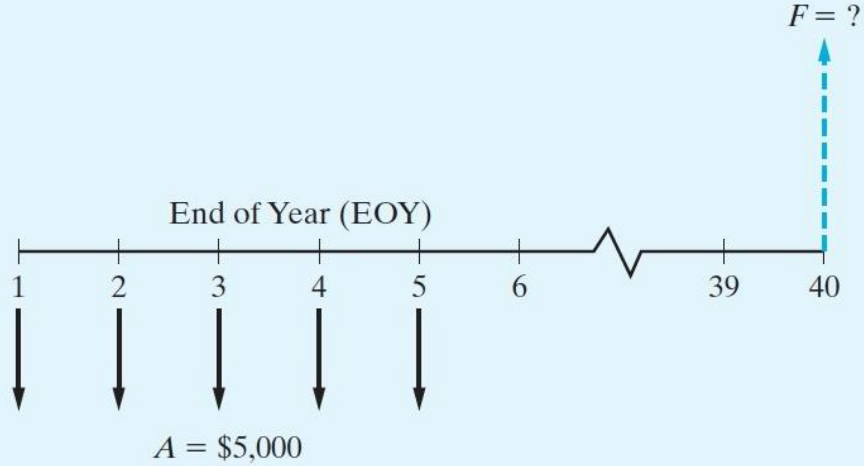
Ex 4-15

Solution

The following cash-flow diagram clarifies the timing of the cash flows for the original 401(k) investment plan.



The easiest way to approach this is to first find the future equivalent of the annuity as of time 5.

$$F_5 = \$5{,}000\,(F/A,\,8\%,\,5) = \$5{,}000\,(5.8666) = \$29{,}333.$$

To determine F_{40}, F_5 can now be denoted P_5, and

$$F_{40} = P_5\,(F/P,\,8\%,\,35) = \$29{,}333\,(14.7853) = \$433{,}697.$$

4.10 EQUIVALENCE CALCULATIONS INVOLVING MULTIPLE INTEREST RATES

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- This section provides examples involving two or more equivalence calculations to solve for an unknown quantity.
- The EOY cash flow convention is used
- The interest rate is constant over the N time periods

4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-16

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EXAMPLE 4-16 **Calculating Equivalent *P*, *F*, and *A* Values**

VIDEO SOLUTIONS

Figure 4-10 depicts an example problem with a series of year-end cash flows extending over eight years. The amounts are \$100 for the first year, \$200 for the second year, \$500 for the third year, and \$400 for each year from the fourth through the eighth. These could represent something like the expected maintenance expenditures for a certain piece of equipment or payments into a fund. Note that the payments are shown at the end of each year, which is a standard assumption (convention) for this book and for economic analyses in general, unless we have information to the contrary. It is desired to find

(a) the present equivalent expenditure, P_0;

(b) the future equivalent expenditure, F_8;

(c) the annual equivalent expenditure, A

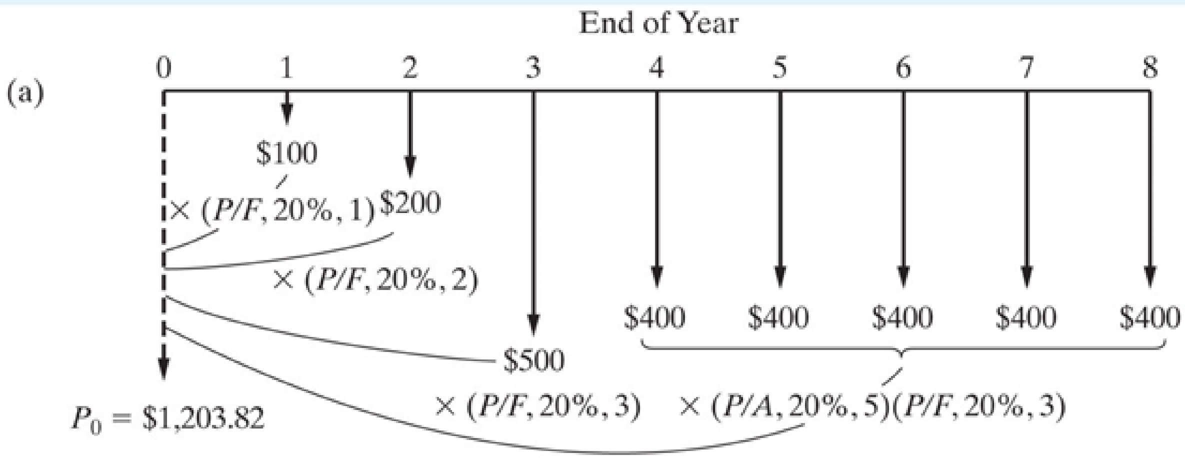
of these cash flows if the annual interest rate is 20%. Solve by hand and by using a spreadsheet.

4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-16

Solution by Hand

(a) To find the equivalent P_0, we need to sum the equivalent values of all payments as of the beginning of the first year (time zero). The required movements of money through time are shown graphically in Figure 4-10(a).



EXAMPLE 4-16 Calculating Equivalent *P*, *F*, and *A* Values

$$
\begin{aligned}
P_0 &= F_1(P/F, 20\%, 1) && = \$100(0.8333) && = \$83.33 \\
&\quad + F_2(P/F, 20\%, 2) && \quad + \$200(0.6944) && \quad + 138.88 \\
&\quad + F_3(P/F, 20\%, 3) && \quad + \$500(0.5787) && \quad + 289.35 \\
&\quad + A(P/A, 20\%, 5) \times (P/F, 20\%, 3) && \quad + \$400(2.9900) \times (0.5787) && \quad + 692.26 \\
& && && \overline{\$1{,}203.82.}
\end{aligned}
$$

Ex 4-16

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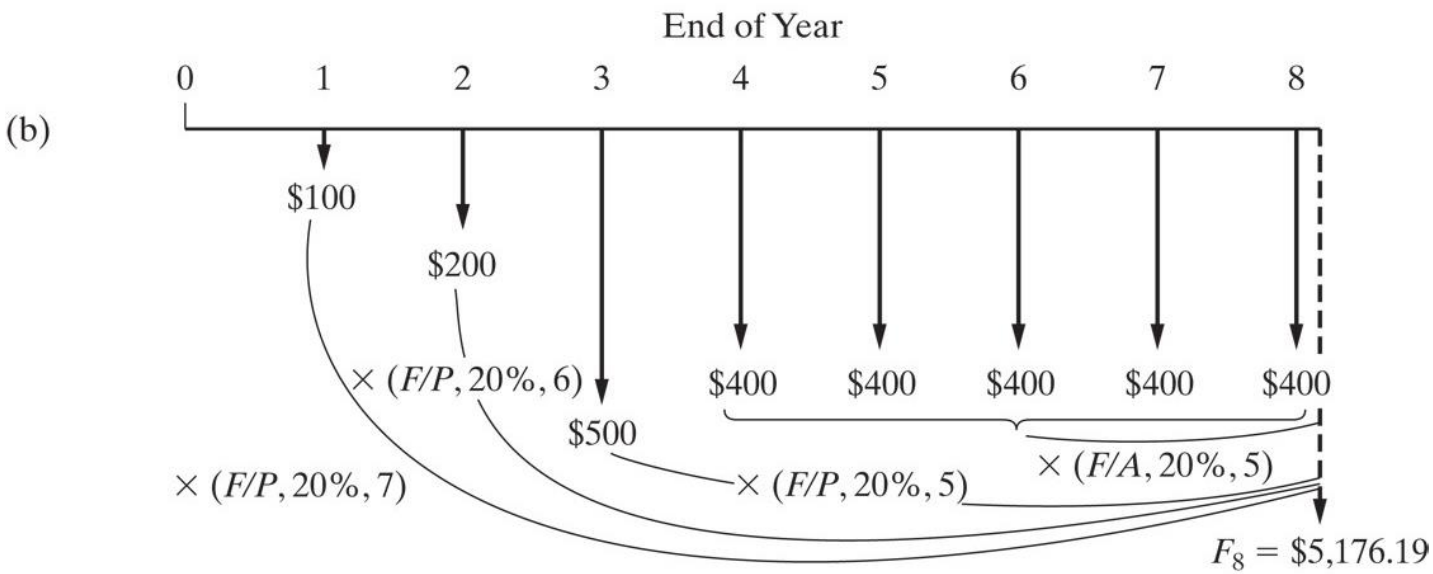
(b) To find the equivalent F_8, we can sum the equivalent values of all payments as of the end of the eighth year (time eight). Figure 4-10(b) indicates these



movements of money through time. However, since the equivalent P_0 is already known to be \$1,203.82, we can directly calculate

$$F_8 = P_0(F/P, 20\%, 8) = \$1,203.82(4.2998) = \$5,176.19.$$

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Ex 4-16

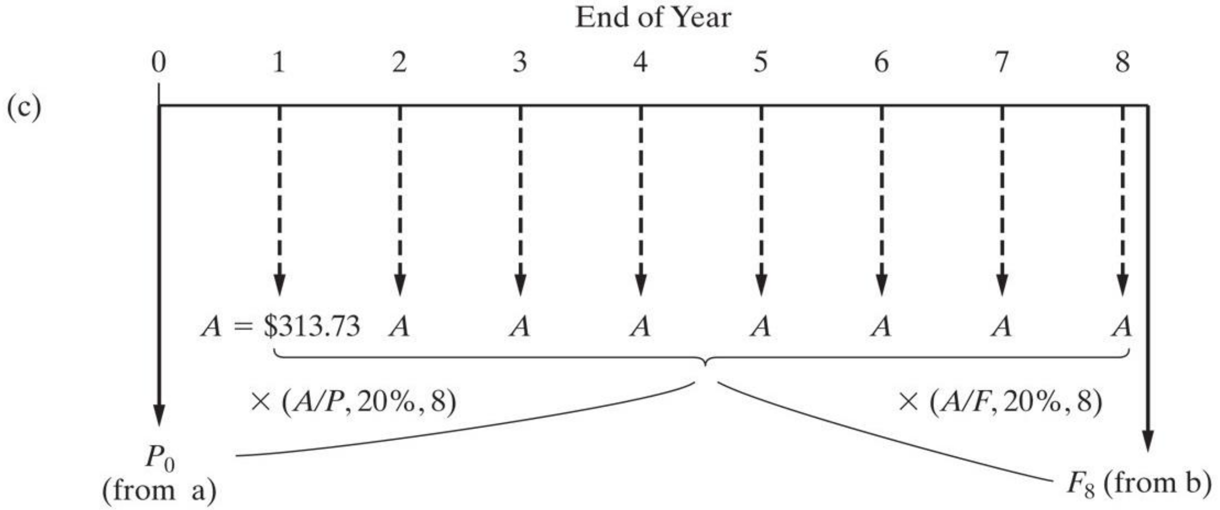
(c) The equivalent A of the irregular cash flows can be calculated directly from either P_0 or F_8 as

$$A = P_0(A/P, 20\%, 8) = \$1{,}203.82(0.2606) = \$313.73$$

or

$$A = F_8(A/F, 20\%, 8) = \$5{,}176.19(0.0606) = \$313.73.$$

The computation of A from P_0 and F_8 is shown in Figure 4-10(c). Thus, we find that the irregular series of payments shown in Figure 4-10 is equivalent to \$1,203.82 at time zero, \$5,176.19 at time eight, or a uniform series of \$313.73 at the end of each of the eight years.



4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-16

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Spreadsheet Solution

Figure 4-11 displays a spreadsheet solution for this example. The present equivalent (P_0) of the tabulated cash flows is easily computed by using the NPV function with the stated interest rate (20% in cell B1). The future equivalent (F_8) is determined from the present equivalent by using the (F/P, $i\%$, N) relationship. The annual equivalent is also determined from the present equivalent by applying the PMT function. The slight differences in results when compared to the hand solution are due to rounding of the interest factor values in the hand solution.

i/y	20%		EOY	Cash Flow	
years in series	8		0	0	
			1	100	
			2	200	
			3	500	
			4	400	
			5	400	
			6	400	
			7	400	
			8	400	
P0	$1,204	$= NPV(rate, value1, [value\ 2], ..)$			
F8	5176	$= P_0 \cdot (1+i)^N$			
A	$314	$= PMT(rate, N, -P0)$			

4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-17

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EXAMPLE 4-17 **How Much is that Last Payment? (Example 4-12 Revisited)**

In Example 4-12, we looked at paying off a loan early by increasing the annual payment. The $100,000 loan was to be repaid in 30 annual installments of $8,880 at an interest rate of 8% per year. As part of the example, we determined that the loan could be paid in full after 21 years if the annual payment was increased to $10,000.

As with most real-life loans, the final payment will be something different (usually less) than the annuity amount. This is due to the effect of rounding in the interest calculations—you can't pay in fractions of a cent! For this example, determine the amount of the 21st (and final) payment on the $100,000 loan when 20 payments of $10,000 have already been made. The interest rate remains at 8% per year.

4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

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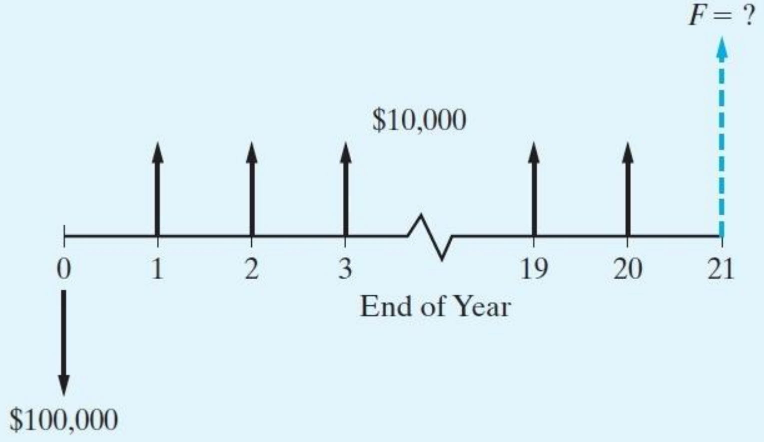
Ex 4-17

Solution

The cash-flow diagram for this example is shown below. It is drawn from the lender's viewpoint.



We need to determine the value of F that will make the present equivalent of all loan payments equal to the amount borrowed. We can do this by discounting all of the payments to time 0 (including the final payment, F) and setting their value equal to $100,000.

$$\$10{,}000\,(P/A,\,8\%,\,20) + F\,(P/F,\,8\%,\,21) = \$100{,}000$$

$$\$10{,}000\,(9.8181) + F\,(0.1987) = \$100{,}000$$

$$F = \$9{,}154.50$$

Thus, a payment of $9,154.50 is needed at the end of year 21 to pay off the loan.

4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-18

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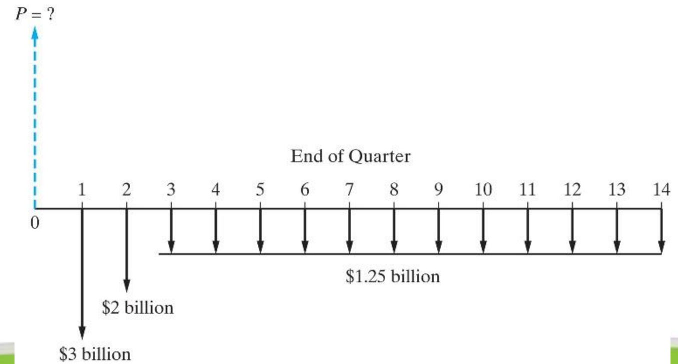
EXAMPLE 4-18 **The Present Equivalent of BP's Payment Schedule**

In this example, we answer the question posed at the beginning of the chapter—what is the present equivalent value of BP's proposed payment schedule? Recall that BP will pay \$3 billion at the end of the third quarter of 2010 and another \$2 billion in the fourth quarter of 2010. Twelve additional payments of \$1.25 billion each quarter thereafter will result in a total of \$20 billion having been paid into the fund. The interest rate is 3% per quarter.

Solution

Figure 4-12 shows the cash-flow diagram for this situation. The present equivalent of the cash flows is

$$\begin{aligned} P &= \$3 \text{ billion } (P/F, 3\%, 1) + \$2 \text{ billion } (P/F, 3\%, 2) \\ &\quad + \$1.25 \text{ billion } (P/A, 3\%, 12)\,(P/F, 3\%, 2) \\ &= \$3 \text{ billion } (0.9709) + \$2 \text{ billion } (0.9426) + \$1.25\,(9.9540)(0.9426) \\ &= \$16.53 \text{ billion.} \end{aligned}$$



4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-19

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EXAMPLE 4-19 **Determining an Unknown Annuity Amount**

Two receipts of \$1,000 each are desired at the EOYs 10 and 11. To make these receipts possible, four EOY annuity amounts will be deposited in a bank at EOYs 2, 3, 4, and 5. The bank's interest rate (i) is 12% per year.

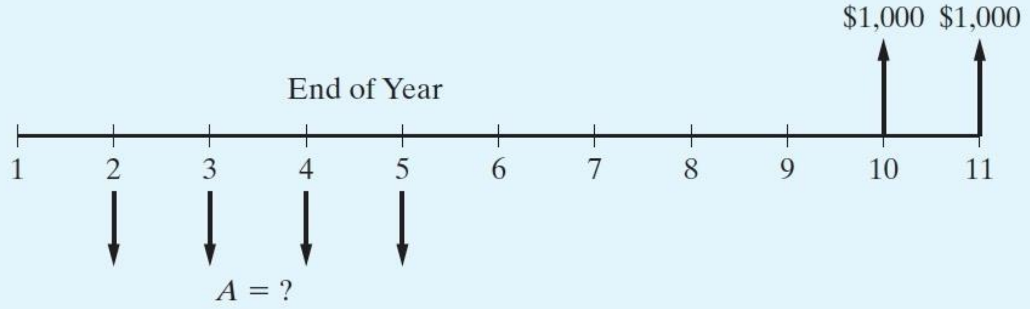
(a) Draw a cash-flow diagram for this situation.

(b) Determine the value of A that establishes equivalence in your cash-flow diagram.

(c) Determine the lump-sum value at the end of year 11 of the completed cash-flow diagram based on your answers to Parts (a) and (b).

Solution

(a) Cash-flow diagrams can make a seemingly complex problem much clearer. The cash-flow diagram for this example is shown below.



4.9 DEFERRED ANNUITIES (UNIFORM SERIES) دفعات مؤجلة

Ex 4-19

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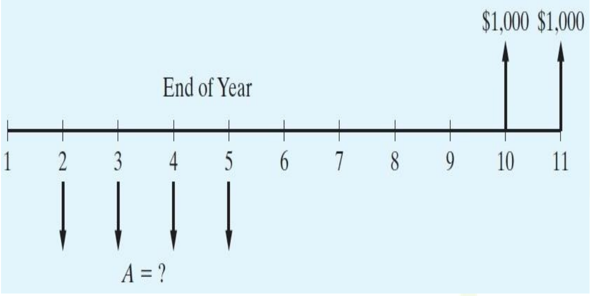


Determining an Unknown Annuity Amount

(b) Because the unknown annuity, A, begins at EOY two, it makes sense to establish our reference year for the equivalence calculations at EOY one (remember that the first annuity amount follows its P-equivalent amount by one year). So the P-equivalent at EOY 1 of the four A amounts is

$$P_1 = A(P/A, 12\%, 4).$$

Next we calculate the EOY one P-equivalent of \$1,000 at EOY 10 and \$1,000 at EOY 11 as follows:

$$P_1' = \$1{,}000\,(P/A, 12\%, 2)\,(P/F, 12\%, 8).$$

The $(P/F, 12\%, 8)$ factor is needed to discount the equivalent value of the A amounts at EOY nine to EOY one. By equating both P-equivalents at EOY one, we can solve for the unknown amount, A.

$$P_1 = P_1'$$

$$A\,(P/A, 12\%, 4) = \$1{,}000\,(P/A, 12\%, 2)\,(P/F, 12\%, 8),$$

or

$$3.0373\,A = \$682.63$$

and

$$A = \$224.75.$$

Therefore, we conclude that deposits of \$224.75 at EOYs two, three, four, and five are equivalent to \$1,000 at EOYs 10 and 11 if the interest rate is 12% per year.

Ex 4-19

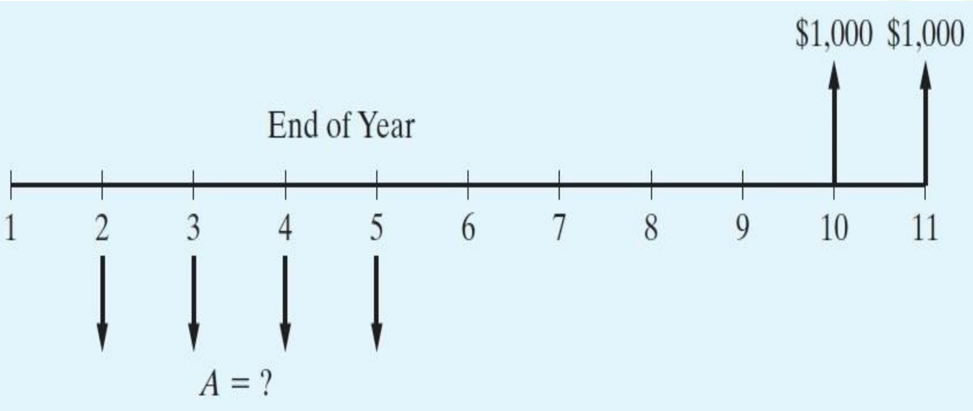
EXAMPLE 4-19 Determining an Unknown Annuity Amount

(c) Now we need to calculate the F-equivalent at time 11 of the −\$224.75 annuity in years 2 through 5 and the \$1,000 annuity in years 10 and 11.

$$-\$224.75\,(F/A, 12\%, 4)\,(F/P, 12\%, 6) + \$1{,}000\,(F/A, 12\%, 2)$$

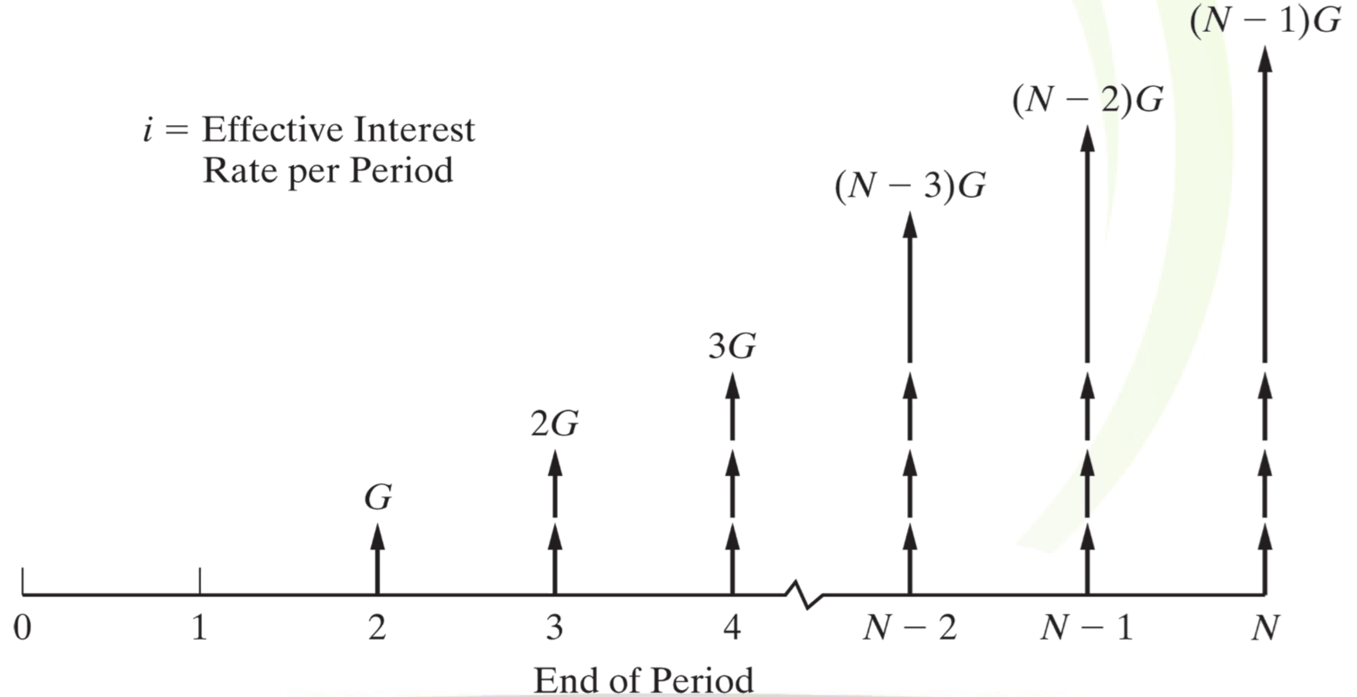
$$= -\$0.15$$

This value should be zero, but round-off error in the interest factors causes a small difference of \$0.15.



4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

- Used for problems involve receipts or expenses that are projected to increase or decrease by a uniform amount each period, i.e. arithmetic sequence of cash flows.
- Ex: equipment maintenance relative to purchasing the equipment may increase by a roughly a constant amount each year
- G: uniform gradient amount: 0G, 1G, 2G, …, (N-1)G
- The first uniform gradient occurs at end of period 2



4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

Finding P when given G

$$P = G\left(\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right)$$

$$P = G(P/G, i\%, N)$$

4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

Finding A when given G

$$A = G[\frac{1}{i} - \frac{N}{(1+i)^N - 1}]$$

$$A = P(A/P, i\%, N)$$

$$A = G(A/G, i\%, N)$$

Finding F when given G

$$F = G(\frac{1}{i}[\frac{(1+i)^N - 1}{i} - N]$$

$$F = P(F/P, i\%, N)$$

4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

Ex 4-20

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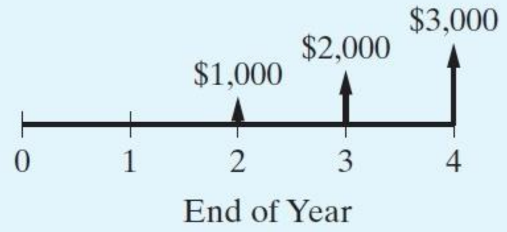
EXAMPLE 4-20 **Using the Gradient Conversion Factors to Find *P* and *A***

As an example of the straightforward use of the gradient conversion factors, suppose that certain EOY cash flows are expected to be $1,000 for the *second* year, $2,000 for the third year, and $3,000 for the fourth year and that, if interest is 15% per year, it is desired to find

(a) present equivalent value at the beginning of the first year,

(b) uniform annual equivalent value at the end of each of the four years.

Solution



4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

Ex 4-20

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EXAMPLE 4-20 **Using the Gradient Conversion Factors to Find *P* and *A***

Observe that this schedule of cash flows fits the model of the arithmetic gradient formulas with $G = \$1{,}000$ and $N = 4$. Note that there is no cash flow at the end of the first period.

(a) The present equivalent can be calculated as

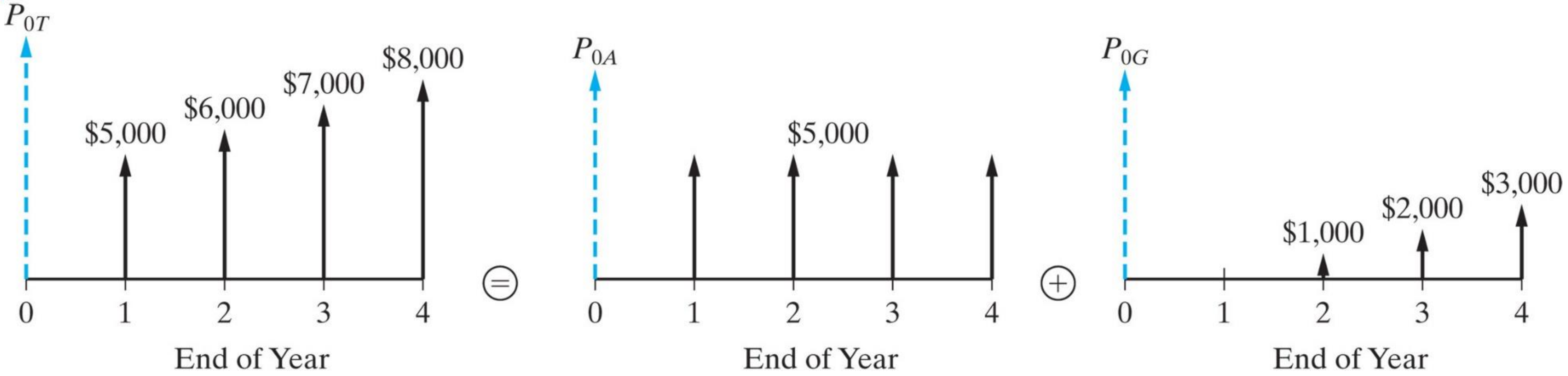
$$P_0 = G(P/G, 15\%, 4) = \$1{,}000(3.79) = \$3{,}790.$$

(b) The annual equivalent can be calculated from Equation (4-26) as

$$A = G(A/G, 15\%, 4) = \$1{,}000(1.3263) = \$1{,}326.30.$$

Of course, once P_0 is known, the value of A can be calculated as

$$A = P_0(A/P, 15\%, 4) = \$3{,}790(0.3503) = \$1{,}326.30.$$



Ex 4-21

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EXAMPLE 4-21 **Present Equivalent of an Increasing Arithmetic Gradient Series**

As a further example of the use of arithmetic gradient formulas, suppose that we have cash flows as follows:

End of Year	Cash Flows ($)
1	5,000
2	6,000
3	7,000
4	8,000

Also, assume that we wish to calculate their present equivalent at $i = 15\%$ per year, using gradient conversion factors.

Ex 4-21

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EXAMPLE 4-21 **Present Equivalent of an Increasing Arithmetic Gradient Series**

Solution

The schedule of cash flows is depicted in the left-hand diagram of Figure 4-14. The right two diagrams of Figure 4-14 show how the original schedule can be broken into two separate sets of cash flows, an annuity series of \$5,000 plus an arithmetic gradient of \$1,000 that fits the general gradient model for which factors are tabled. The summed present equivalents of these two separate sets of cash flows equal the present equivalent of the original problem. Thus, using the symbols shown in Figure 4-14, we have

$$P_{0T} = P_{0A} + P_{0G}$$

$$= A(P/A, 15\%, 4) + G(P/G, 15\%, 4)$$

$$= \$5{,}000(2.8550) + \$1{,}000(3.79) = \$14{,}275 + 3{,}790 = \$18{,}065.$$

The annual equivalent of the original cash flows could be calculated with the aid of Equation (4-26) as follows:

$$A_T = A + A_G$$

$$= \$5{,}000 + \$1{,}000(A/G,\ 15\%,\ 4) = \$6{,}326.30.$$

A_T is equivalent to P_{0T} because $\$6{,}326.30(P/A, 15\%, 4) = \$18{,}061$, which is the same value obtained previously (subject to round-off error).

4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

Ex 4-22

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EXAMPLE 4-22 **Present Equivalent of a Decreasing Arithmetic Gradient Series**

VIDEO SOLUTIONS

For another example of the use of arithmetic gradient formulas, suppose that we have cash flows that are timed in exact reverse of the situation depicted in Example 4-21. The left-hand diagram of Figure 4-15 shows the following sequence of cash flows:

End of Year	Cash Flows ($)
1	8,000
2	7,000
3	6,000
4	5,000

Calculate the present equivalent at $i = 15\%$ per year, using arithmetic gradient interest factors.

Solution

The right two diagrams of Figure 4-15 show how the uniform gradient can be broken into two separate cash-flow diagrams. In this example, we are *subtracting* an arithmetic gradient of $1,000 from an annuity series of $8,000.

4.11 UNIFORM (ARITHMETIC) GRADIENT OF CASH FLOWS

Ex 4-22

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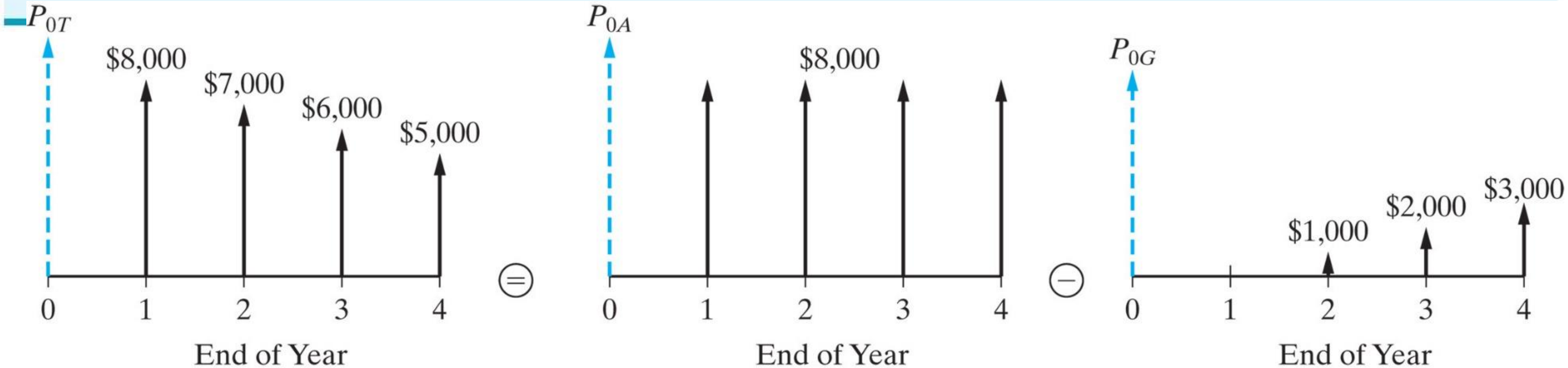
EXAMPLE 4-22 **Present Equivalent of a Decreasing Arithmetic Gradient Series**

So,

$$
\begin{aligned}
P_{0T} &= P_{0A} - P_{0G} \\
&= A(P/A, 15\%, 4) - G(P/G, 15\%, 4) \\
&= \$8{,}000(2.8550) - \$1{,}000(3.79) \\
&= \$22{,}840 - \$3{,}790 = \$19{,}050.
\end{aligned}
$$

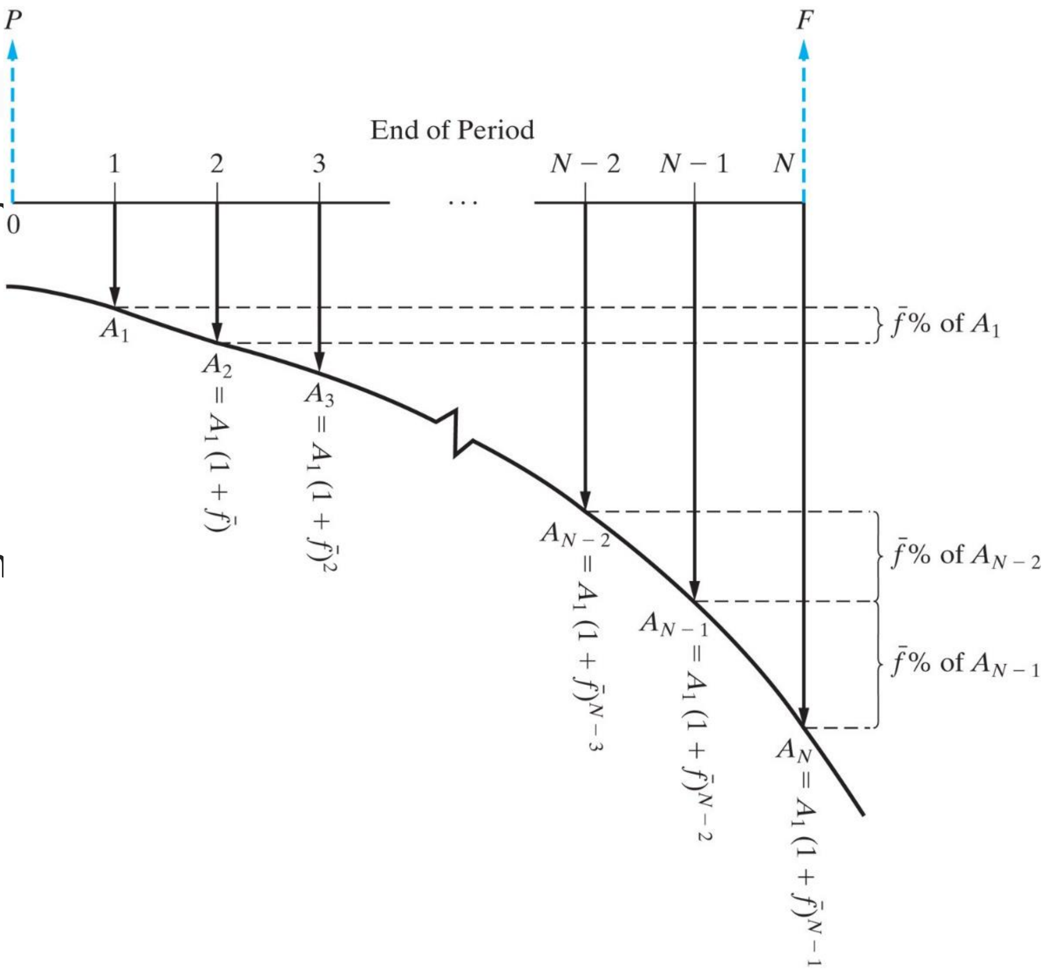
Again, the annual equivalent of the original decreasing series of cash flows can be calculated by the same rationale:

$$
\begin{aligned}
A_T &= A - A_G \\
&= \$8{,}000 - \$1{,}000(A/G, 15\%, 4) \\
&= \$6{,}673.70.
\end{aligned}
$$



4.12 GEOMETRIC SEQUENCES OF CASH FLOWS

- Used for some economic equivalence problems involve projected cash flow patterns changing at an average rate, $\bar{f}$, each period
- Ex: a fixed amount of a commodity that inflates in price at a constant rate each year
- The resultant EOY cash-flow pattern is referred to as a ***geometric gradient series***
- $A_k = (A_{k-1})(1+\bar{f}), 2 \leq k \leq N$
- $A_N = (A_1)(1+\bar{f})^{N-1}$
- $\bar{f} = \frac{A_k - A_{k-1}}{A_{k-1}}$



4.12 GEOMETRIC SEQUENCES OF CASH FLOWS

- $A_k = (A_{k-1})(1+\bar{f}), 2 \le k \le N$
- $A_N = (A_1)(1+\bar{f})^{N-1}$
- $\bar{f} = \frac{A_k - A_{k-1}}{A_{k-1}}$
- $P = \begin{cases} \frac{A_1[1-(1+i)^{-N}(1+\bar{f})^N}{i-\bar{f}}, \bar{f} \neq 1 \\ A_1 N(1+i)^{-1}, \bar{f} = 1 \end{cases}$
- $P = \begin{cases} \frac{A_1[1-(P/F,i\%,N)(F/P,\bar{f}\%,N)]}{i-\bar{f}}, \bar{f} \neq 1 \\ A_1 N(P/F,i\%,1), \bar{f} = 1 \end{cases}$

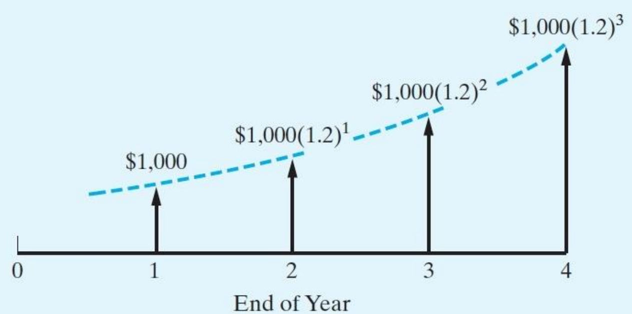
4.12 GEOMETRIC SEQUENCES OF CASH FLOWS

EXAMPLE 4-23 **Equivalence Calculations for an Increasing Geometric Gradient Series**

VIDEO SOLUTIONS

Consider the following EOY geometric sequence of cash flows and determine the P, A, and F equivalent values. The rate of increase is 20% per year after the first year, and the interest rate is 25% per year.

Solution



$$P = \frac{\$1,000\left[1-(P/F,25\%,4)(F/P,20\%,4)\right]}{0.25-0.20}$$

$$\begin{aligned} P &= \frac{\$1,000}{0.05}\left[1-(0.4096)(2.0736)\right] \\ &= \$20,000(0.15065) \\ &= \$3,013; \end{aligned}$$

$$A = \$3,013(A/P,25\%,4) = \$1,275.70;$$

$$F = \$3,013(F/P,25\%,4) = \$7,355.94.$$

4.12 GEOMETRIC SEQUENCES OF CASH FLOWS

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EXAMPLE 4-24 **Equivalence Calculations for a Decreasing Geometric Gradient Series**

Suppose that the geometric gradient in Example 4-23 begins with \$1,000 at EOY one and *decreases* by 20% per year after the first year. Determine P, A, and F under this condition.

Solution

The value of $\bar{f}$ is -20% in this case. The desired quantities are as follows:

$$P = \frac{\$1{,}000[1 - (P/F, 25\%, 4)(F/P, -20\%, 4)]}{0.25 - (-0.20)}$$

$$= \frac{\$1{,}000}{0.45}\left[1 - (0.4096)(1 - 0.20)^4\right]$$

$$= \$2{,}222.22(0.83222)$$

$$= \$1{,}849.38;$$

$$A = \$1{,}849.38(A/P, 25\%, 4) = \$783.03;$$

$$F = \$1{,}849.38(F/P, 25\%, 4) = \$4{,}515.08.$$

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EXAMPLE 4-26 **A Retirement Savings Plan**

VIDEO SOLUTIONS

On your 23rd birthday you decide to invest \$4,500 (10% of your annual salary) in a mutual fund earning 7% per year. You will continue to make annual deposits equal to 10% of your annual salary until you retire at age 62 (40 years after you started your job). You expect your salary to increase by an average of 4% each year during this time. How much money will you have accumulated in your mutual fund when you retire?

Solution

Since the amount of your deposit is 10% of your salary, each year the amount you deposit will increase by 4% as your salary increases. Thus, your deposits constitute a geometric gradient series with $f = 4\%$ per year. We can use Equation (4-30) to determine the present equivalent amount of the deposits.

$$P = \frac{\$4{,}500[1 - (P/F, 7\%, 40)(F/P, 4\%, 40)]}{0.07 - 0.04}$$

$$= \frac{\$4{,}500[1 - (0.0668)(4.8010)]}{0.03}$$

$$= \$101{,}894.$$

Now the future worth at age 62 can be determined.

$$F = \$101{,}894(F/P, 7\%, 40)$$

$$= \$101{,}894(14.9745)$$

$$= \$1{,}525{,}812.$$

This savings plan will make you a millionaire when you retire. *Moral:* Start saving early!

4.13 INTEREST RATE VARY WITH TIME

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- Interest rates often change with time (e.g., a variable rate mortgage).
- We often must resort to moving cash flows one period at a time, reflecting the interest rate for that single period.

4.13 INTEREST RATE VARY WITH TIME

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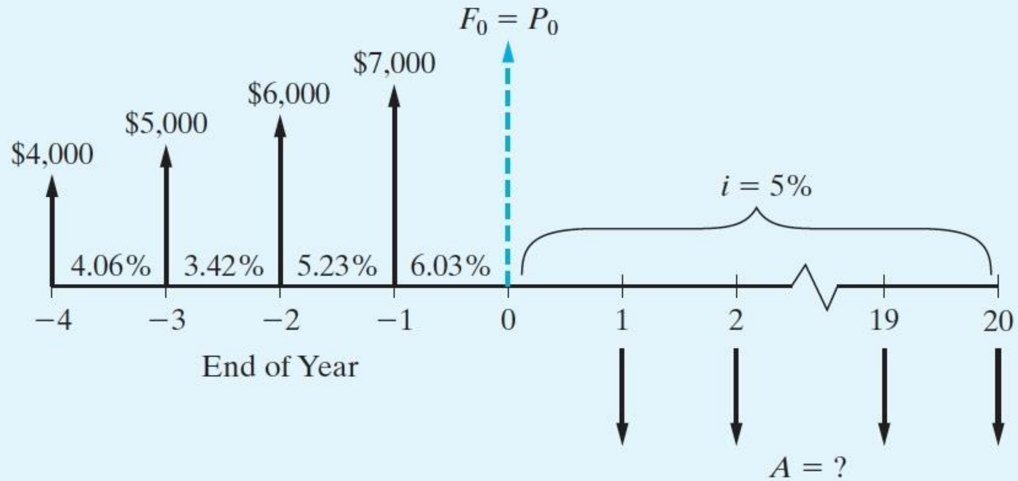
EXAMPLE 4-27 **Compounding with Changing Interest Rates**

VIDEO SOLUTIONS

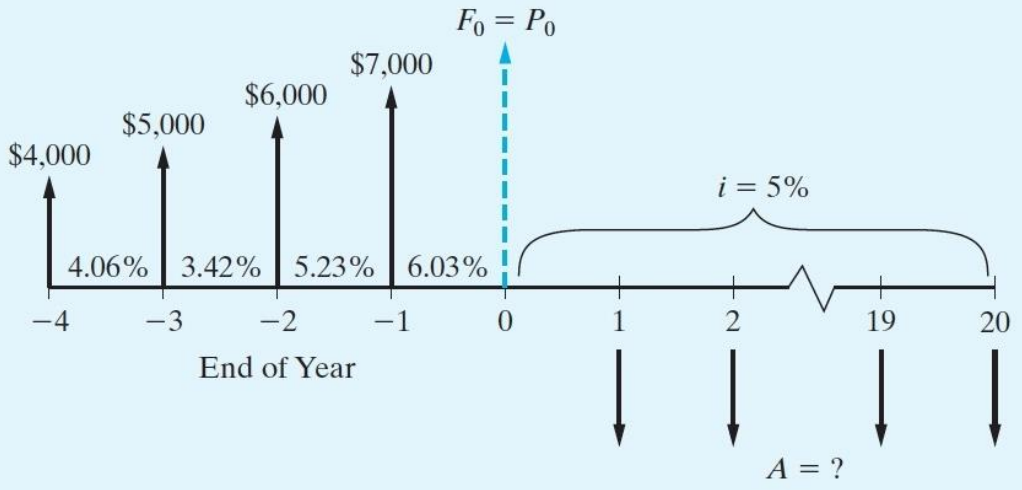
Ashea Smith is a 22-year-old senior who used the Stafford loan program to borrow \$4,000 four years ago when the interest rate was 4.06% per year. \$5,000 was borrowed three years ago at 3.42%. Two years ago she borrowed \$6,000 at 5.23%, and last year \$7,000 was borrowed at 6.03% per year. Now she would like to consolidate her debt into a single 20-year loan with a 5% fixed annual interest rate. If Ashea makes annual payments (starting in one year) to repay her total debt, what is the amount of each payment?

Solution

The following cash-flow diagram clarifies the timing of Ashea's loans and the applicable interest rates. The diagram is drawn using Ashea's viewpoint.



4.13 INTEREST RATE VARY WITH TIME



Before we can find the annual repayment amount, we need to find the current (time 0) equivalent value of the four loans. This problem can be solved by compounding the amount owed at the beginning of each year by the interest rate that applies to each individual year and repeating this process over the four years to obtain the total current equivalent value.

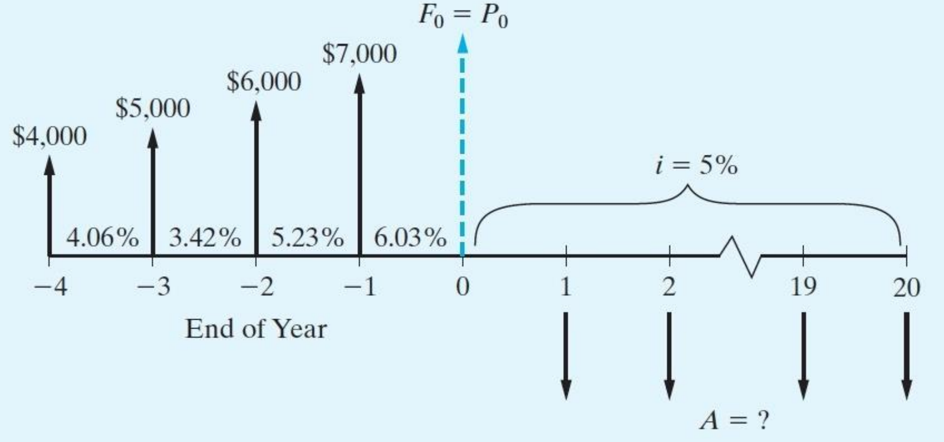
$$F_{-3} = \$4{,}000(F/P, 4.06\%, 1) + \$5{,}000 = \$4{,}000(1.0406) + \$5{,}000 = \$9{,}162.40$$

$$F_{-2} = \$9{,}162.40(F/P, 3.42\%, 1) + \$6{,}000 = \$15{,}475.75$$

$$F_{-1} = \$15{,}475.75(F/P, 5.23\%, 1) + \$7{,}000 = \$23{,}285.13$$

$$F_0 = \$23{,}285.13(F/P, 6.03\%, 1) = \$24{,}689.22$$

4.13 INTEREST RATE VARY WITH TIME



$$F_{-3} = \$4{,}000(F/P, 4.06\%, 1) + \$5{,}000 = \$4{,}000(1.0406)$$

$$+ \$5{,}000 = \$9{,}162.40$$

$$F_{-2} = \$9{,}162.40(F/P, 3.42\%, 1) + \$6{,}000 = \$15{,}475.75$$

$$F_{-1} = \$15{,}475.75(F/P, 5.23\%, 1) + \$7{,}000 = \$23{,}285.13$$

$$F_0 = \$23{,}285.13(F/P, 6.03\%, 1) = \$24{,}689.22$$

Notice that it was a simple matter to substitute $(F/P, i\%, N) = (1 + i)^N$ for the noninteger values of i.

Now that we have the current equivalent value of the amount Ashea borrowed ($F_0 = P_0$), we can easily compute her annual repayment amount over 20 years when the interest rate is fixed at 5% per year.

$$A = \$24{,}689.22(A/P, 5\%, 20) = \$24{,}689.22(0.0802) = \$1{,}980.08 \text{ per year}$$

4.14 WHEN INTEREST RATES VARY WITH TIME DIFFERENT PROCEDURES ARE NECESSARY.

- The present equivalent of a cash flow occurring at the end of period N can be computed with the equation below, where i_k is the interest rate for the k^th period.
- $P = \frac{F_N}{\prod_{k=1}^{N}(1+i_k)}$
- If F_4 = \$2,500 and i_1=8%, i_2=10%, and i_3=11%, then

$$P = \$2,500(P/F, 8\%, 1)(P/F, 10\%, 1)(P/F, 11\%, 1)$$

$$P = \$2,500(0.9259)(0.9091)(0.9009) = \$1,896$$

4.14 NOMINAL AND EFFECTIVE INTEREST RATE

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- More often than not, the time between successive compounding, or the interest period, is less than one year (e.g., daily, monthly, quarterly).
- **The annual rate is also known as:**
  - ***nominal* rate**
  - **Annual Percentage Rate (APR)**
- A *nominal* rate (r) of 12%, compounded monthly, means an interest of 1% (12%/12) would accrue (accumulate) each month, and the annual rate would be *effectively* somewhat greater than 12%.
- The more frequent the compounding the greater the *effective* interest.
- Let $\boldsymbol{r}$ be the nominal, annual interest rate and $\boldsymbol{M}$ the number of compounding periods per year. We can find, $\boldsymbol{i}$, the effective interest:
- $\boldsymbol{i = (1 + \frac{r}{M})^{M} - 1}$
- For an 18% nominal rate, compounded quarterly, the effective interest is.
- $i = (1 + \frac{0.18}{4})^{4} - 1 = 19.25\%$

4.14 NOMINAL AND EFFECTIVE INTEREST RATE

Consider \$1,000 to be invested for 3 years at a nominal interest rate of 12% compounded Semiannually (r = 12%, M = 2)

Interest during the first 6 months:

$$\$1,000 \cdot (\frac{0.12}{2}) = \$60$$

Total principal and interest at the beginning of the second six months:

$$P + Pi = \$1,000 + \$60 = \$1,060$$

Interest during the second 6 months:

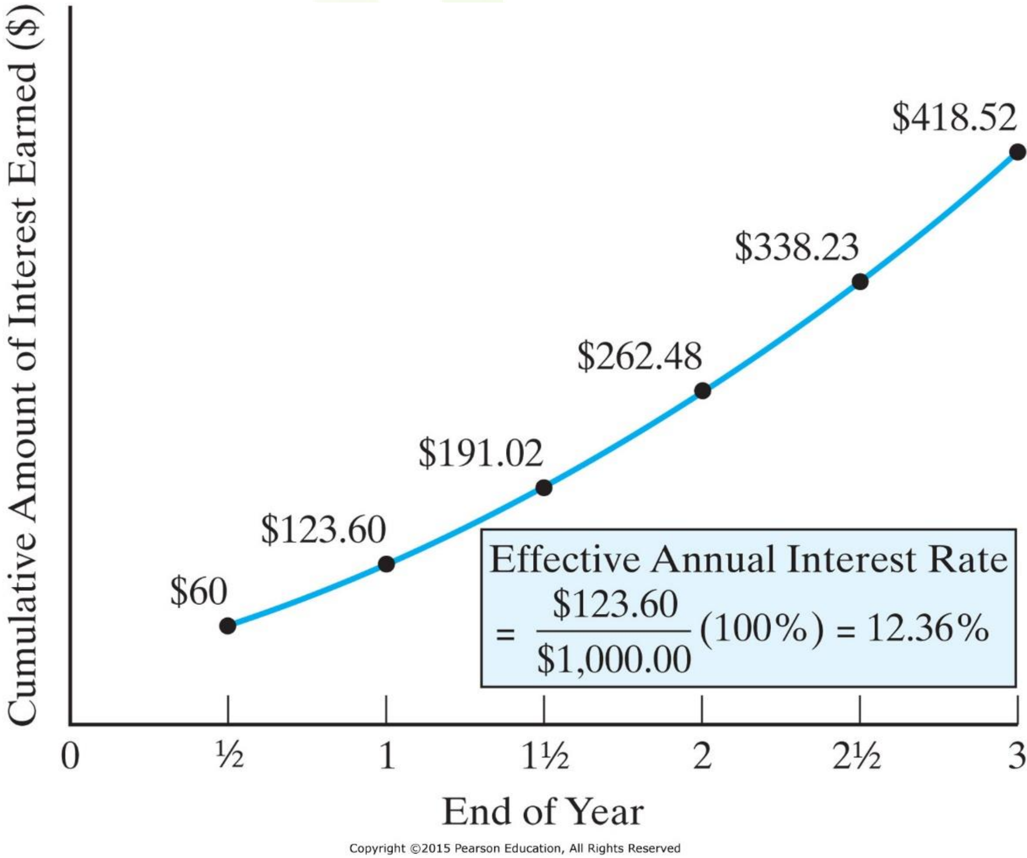
$$\$1,060 \cdot \left(\frac{0.12}{2}\right) = \$63.60$$

Total interest during the year:

$$\$60.00 + \$63.60 = \$123.60$$

The effective annual interest rate

$$\frac{\$123.60}{1,000} \cdot 100\% = 12.36\%$$



4.14 NOMINAL AND EFFECTIVE INTEREST RATE

TABLE 4-4 Effective Interest Rates for Various Nominal Rates and Compounding Frequencies

Compounding Frequency	Number of Compounding Periods per Year, *M*	Effective Rate (%) for Nominal Rate of 6%	8%	10%	12%	15%	24%
Annually	1	6.00	8.00	10.00	12.00	15.00	24.00
Semiannually	2	6.09	8.16	10.25	12.36	15.56	25.44
Quarterly	4	6.14	8.24	10.38	12.55	15.87	26.25
Bimonthly	6	6.15	8.27	10.43	12.62	15.97	26.53
Monthly	12	6.17	8.30	10.47	12.68	16.08	26.82
Daily	365	6.18	8.33	10.52	12.75	16.18	27.11

4.15 COMPOUNDING MORE THAN ONCE A YEAR

Single amounts

- If a nominal interest rate is quoted and the number of compounding periods per year, and number of years are known, use equations, respectively:
- $F = P(1+i)^N$
- $\boldsymbol{i = (1+\frac{r}{M})^{M} - 1}$

EXAMPLE 4-29 Future Equivalent when Interest Is Compounded Quarterly

Suppose that a \$100 lump-sum amount is invested for 10 years at a nominal interest rate of 6% compounded quarterly. How much is it worth at the end of the 10th year?

Solution

There are four compounding periods per year, or a total of $4 \times 10 = 40$ interest periods. The interest rate per interest period is $6\%/4 = 1.5\%$. When the values are used in Equation (4-3), one finds that

$$F = P(F/P, 1.5\%, 40) = \$100.00(1.015)^{40} = \$100.00(1.814) = \$181.40.$$

Alternatively, the effective interest rate from Equation (4-32) is 6.14%. Therefore,

$$F = P(F/P, 6.14\%, 10) = \$100.00(1.0614)^{10} = \$181.40.$$

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EXAMPLE 4-30 Computing a Monthly Auto Payment

Stan Moneymaker has a bank loan for \$10,000 to pay for his new truck. This loan is to be repaid in equal *end-of-month* installments for five years with a nominal interest rate of 12% compounded monthly. What is the amount of each payment?

Solution

The number of installment payments is $5 \times 12 = 60$, and the interest rate per month is $12\%/12 = 1\%$. When these values are used in Equation (4-15), one finds that

$$A = P(A/P, 1\%, 60) = \$10{,}000(0.0222) = \$222.$$

Notice that there is a cash flow at the end of each month (interest period), including month 60, in this example.

Uniform and Gradient series

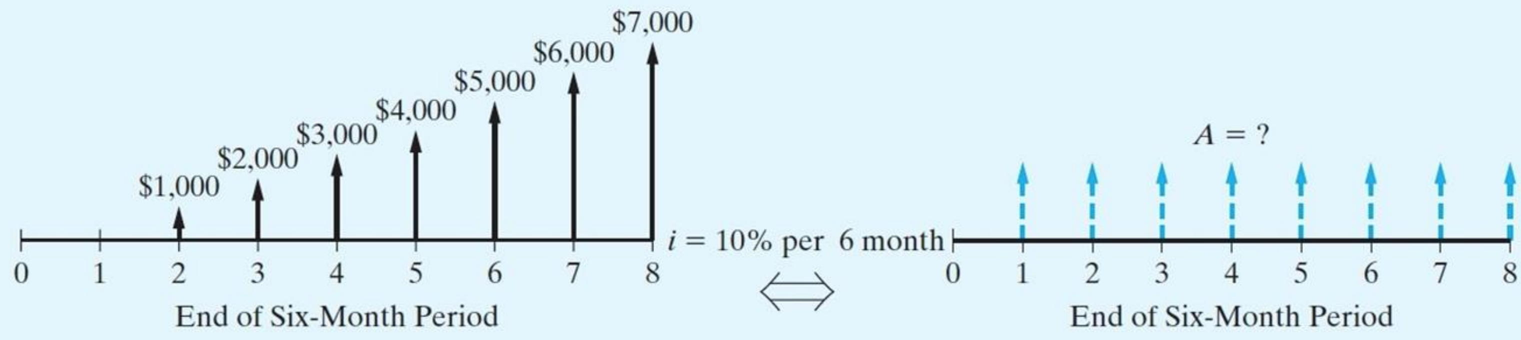
EXAMPLE 4-31 **Uniform Gradient Series and Semiannual Compounding**

Certain operating savings are expected to be 0 at the end of the first six months, to be \$1,000 at the end of the second six months, and to increase by \$1,000 at the end of each six-month period thereafter, for a total of four years. It is desired to find the equivalent uniform amount, A, at the end of each of the eight six-month periods if the nominal interest rate is 20% compounded semiannually.

Solution

A cash-flow diagram is given below, and the solution is

$$A = G(A/G, 10\%, 8) = \$1{,}000(3.0045) = \$3{,}004.50.$$



The symbol "⟺" in between the cash-flow diagrams indicates that the left-hand cash-flow diagram is *equivalent to* the right-hand cash-flow diagram when the correct value of A has been determined. In Example 4-31, the interest rate per six-month period is 10%, and cash flows occur every six months.

Uniform and Gradient series

EXAMPLE 4-32 Finding the Interest Rate on a Loan

A loan of \$15,000 requires monthly payments of \$477 over a 36-month period of time. These payments include both principal and interest.

(a) What is the nominal interest rate (APR) for this loan?
(b) What is the effective interest rate per year?
(c) Determine the amount of unpaid loan principal after 20 months.

Solution

(a) We can set up an equivalence relationship to solve for the unknown interest rate since we know that $P = \$15{,}000$, $A = \$477$, and $N = 36$ months.

$$\$477 = \$15{,}000(A/P, i_{mo}, 36)$$

$$(A/P,\ i_{mo}, 36) = 0.0318$$

We can now look through Appendix C to find values of i that have an $(A/P,\ i,\ 36)$ value close to 0.0318. From Table C-3 ($i = 3/4\%$), we find $(A/P, 3/4\%, 36) = 0.0318$. Therefore,

$$i_{mo} = 0.75\% \text{ per month}$$

and

$$r = 12 \times 0.75\% = 9\% \text{ per year, compounded monthly.}$$

4.15 COMPOUNDING MORE THAN ONCE A YEAR

Uniform and Gradient series

Co-funded by the Erasmus+ Programme of the European Union

EXAMPLE 4-32 **Finding the Interest Rate on a Loan**

A loan of \$15,000 requires monthly payments of \$477 over a 36-month period of time. These payments include both principal and interest.

(a) What is the nominal interest rate (APR) for this loan?

(b) What is the effective interest rate per year?

(c) Determine the amount of unpaid loan principal after 20 months.

(b) Using Equation (4-32),

$$i_{\text{eff}} = \left(1 + \frac{0.09}{12}\right)^{12} - 1 = 0.0938 \text{ or } 9.38\% \text{ per year.}$$

(c) We can find the amount of the unpaid loan principal after 20 months by finding the equivalent value of the remaining 16 monthly payments as of month 20.

$$P_{20} = \$477(P/A, 3/4\%, 16) = \$477(15.0243) = \$7{,}166.59$$

After 20 payments have been made, almost half of the original principal amount remains. Notice that we used the monthly interest rate of $3/4\%$ in our calculation since the cash flows are occurring monthly.

4.16 CONTINUOUS COMPOUNDING INTEREST FACTORS

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- Interest is typically compounded at the end of discrete periods.
- In most companies cash is always flowing, and should be immediately put to use.
- We can allow compounding to occur continuously throughout the period.
- The effect of this compared to discrete compounding is small in most cases.
- The continuously compounded compound amount factor (single cash flow) at $r\%$ nominal interest for N years is e^{rN}
- e^{rN} *corresponds to* $(1+i)^N$
- We can use the effective interest formula to derive the interest factors.
- $\boldsymbol{i = (1+\frac{r}{M})^{M} - 1}$
- As the number of compounding periods gets larger (M gets larger), we find that
- $i = e^{r} - 1$

4.16 CONTINUOUS COMPOUNDING INTEREST FACTORS

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- Continuous compounding interest factors:

$$(P/F, \underline{\mathrm{r}}\%, N) = e^{-rN}$$

$$(F/A, \underline{\mathrm{r}}\%, N) = \frac{e^{rN} - 1}{e^{r} - 1}$$

$$(P/A, \underline{\mathrm{r}}\%, N) = \frac{e^{rN} - 1}{e^{rN}(e^{r} - 1)}$$

4.16 CONTINUOUS COMPOUNDING INTEREST FACTORS

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TABLE 4-5 Continuous Compounding and Discrete Cash Flows: Interest Factors and Symbols[a]

To Find:	Given:	Factor by which to Multiply "Given"	Factor Name	Factor Functional Symbol
For single cash flows:				
F	P	e^{rN}	Continuous compounding compound amount (single cash flow)	$(F/P, \underline{r}\%, N)$
P	F	e^{-rN}	Continuous compounding present equivalent (single cash flow)	$(P/F, \underline{r}\%, N)$
For uniform series (annuities):				
F	A	$\frac{e^{rN}-1}{e^{r}-1}$	Continuous compounding compound amount (uniform series)	$(F/A, \underline{r}\%, N)$
P	A	$\frac{e^{rN}-1}{e^{rN}(e^{r}-1)}$	Continuous compounding present equivalent (uniform series)	$(P/A, \underline{r}\%, N)$
A	F	$\frac{e^{r}-1}{e^{rN}-1}$	Continuous compounding sinking fund	$(A/F, \underline{r}\%, N)$
A	P	$\frac{e^{rN}(e^{r}-1)}{e^{rN}-1}$	Continuous compounding capital recovery	$(A/P, \underline{r}\%, N)$

[a] $\underline{r}$, nominal annual interest rate, compounded continuously; N, number of periods (years); A, annual equivalent amount (occurs at the end of each year); F, future equivalent; P, present equivalent.

4.16 CONTINUOUS COMPOUNDING INTEREST FACTORS

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EXAMPLE 4-33 **Continuous Compounding and Single Amounts**

You have \$10,000 to invest for two years. Your bank offers 5% interest, compounded continuously for funds in a money market account. Assuming no additional deposits or withdrawals, how much money will be in that account at the end of two years?

Solution

$$F = \$10{,}000\ (F/P, \underline{r} = 5\%, 2) = \$10{,}000\ e^{(0.05)(2)} = \$10{,}000\ (1.1052) = \$11{,}052$$

Comment

If the interest rate was 5% compounded annually, the account would have been worth

$$F = \$10{,}000\ (F/P, 5\%, 2) = \$10{,}000\ (1.1025) = \$11{,}025.$$

4.16 CONTINUOUS COMPOUNDING INTEREST FACTORS

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EXAMPLE 4-34 **Continuous Compounding and Annual Payments**

Suppose that one has a present loan of \$1,000 and desires to determine what equivalent uniform EOY payments, A, could be obtained from it for 10 years if the nominal interest rate is 20% compounded continuously ($M = \infty$).

Solution

Here we utilize the formulation

$$A = P(A/P, \underline{r}\%, N).$$

Since the (A/P) factor is not tabled for continuous compounding, we substitute its inverse (P/A), which is tabled in Appendix D. Thus,

$$A = P \times \frac{1}{(P/A, \underline{20}\%, 10)} = \$1{,}000 \times \frac{1}{3.9054} = \$256.$$

Note that the answer to the same problem, with discrete annual compounding ($M = 1$), is

$$A = P(A/P, 20\%, 10)$$

$$= \$1{,}000(0.2385) = \$239.$$

4.16 CONTINUOUS COMPOUNDING INTEREST FACTORS

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EXAMPLE 4-35 **Continuous Compounding and Semiannual Payments**

VIDEO SOLUTIONS

An individual needs \$12,000 immediately as a down payment on a new home. Suppose that he can borrow this money from his insurance company. He must repay the loan in equal payments every six months over the next eight years. The nominal interest rate being charged is 7% compounded continuously. What is the amount of each payment?

Solution

The nominal interest rate per six months is 3.5%. Thus, A each six months is \$12,000($A/P, \underline{r} = 3.5\%, 16$). By substituting terms in Equation (4-38) and then using its inverse, we determine the value of A per six months to be \$997:

$$A = \$12{,}000\left[\frac{1}{(P/A, \underline{r} = 3.5\%, 16)}\right] = \frac{\$12{,}000}{12.038} = \$997.$$

END OF TRAINING

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The end

REFERENCE

- Engineering Economy, by William G. Sullivan, Elin M. Wicks, and James T. Luxhoj, 16th edition, Prentice Hall.